Perverse Sheaves

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1 Review of constructible sheaves and their derived category

Let X be a variety over $\mathbf{C}$. We will consider it in the analytic topology. Fix a field k.

Definition 1.1. A *local system* on X over k is a sheaf $\mathcal{L}$ of k-spaces such that there is an open covering $X = \bigcup_{\alpha \in I} U_\alpha$ with $\mathcal{L}|_{U_\alpha}$ the constant sheaf for all $\alpha \in I$.

Let $\mathrm{Loc}^{\mathrm{ft}}(X, k)$ be the category of local systems where the stalks are finite type k-modules.

Definition 1.2. A *stratification* of X is a finite collection $(X_s)_{s \in \mathcal{I}}$ of disjoint smooth connected locally closed subvarieties, such that $X = \bigcup_{s \in \mathcal{I}} X_s$ and $\overline{X_s} \cap X_t$ is empty or X_t.

Definition 1.3. Let $(X_s)_{s \in \mathcal{I}}$ be a stratification. A sheaf $\mathcal{F} \in \mathrm{Sh}(X, k)$ is *constructible (with respect to $\mathcal{I}$)* if each $\mathcal{F}|_{X_s} \in \mathrm{Loc}^{\mathrm{ft}}(X_s, k)$.

An object $\mathcal{F} \in D^b(X, k)$ is *constructible* if each $\mathcal{H}^k(\mathcal{F})$ is constructible with respect to some stratification $\mathcal{I}$. These form the full subcategory $D^b_{\mathcal{I}}(X, k)$, and $D^b_c(X, k)$ is the full subcategory of constructible over some $\mathcal{I}$.

Let's recall the six functors associated to a map $f : X \to Y$.

- $f^*\mathcal{F}(U)$ is sheafification of $\varinjlim F(V)$ over $V \subset Y$ open with $V \supset f(U)$.
- $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ (note $f^* \dashv f_*$).
- $f_!(\mathcal{F})(U) = \{s \in \mathcal{F}(f^{-1}(U)) : f|_{\mathrm{supp}\ s} : \mathrm{supp}\ s \to U \text{ is proper}\}$.
- $Rf^!$ (define by $f_! \dashv f^!$).
- $\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}(U), \mathcal{G}(U))$.
- $(\mathcal{F} \otimes \mathcal{G})(U)$ is the sheafification of $\mathcal{F}(U) \otimes \mathcal{G}(U)$ (tensor-hom adjunction).

Proposition 1.4. *All six are defined over D^b_c.*

Example. If $h : Y \hookrightarrow X$ is the inclusion of a locally closed subset, then we have easier descriptions of $h_!$ and $h^!$:

$$h_!(\mathcal{F})_x = \begin{cases} \mathcal{F}_x & x \in Y \\ 0 & x \notin Y \end{cases}$$

$$h^!(\mathcal{F})(U) = \varinjlim \{s \in \mathcal{F}(V) : \text{supp } s \subset U\}$$

ranging over $V \subset X$ open and $V \cap \overline{Y} = U$.

Corollary 1.5. *If $j : U \hookrightarrow X$ is an open embedding, then $j^! \cong j^*$.*

Proposition 1.6. *For $h : Y \hookrightarrow X$ we have isomorphisms*

$$\mathcal{F} \to h^! h_! \mathcal{F} \to h^* h_! \mathcal{F}, \quad h^! h_* \mathcal{F} \to h^* h_* \mathcal{F} \to \mathcal{F}$$

Theorem 1.7. *For $i : Z \hookrightarrow X \hookleftarrow U : j$ complementary closed/open embeddings, we have distinguished triangles*

$$j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F}, \quad i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F}$$

Finally, there is a notion of duality.

Definition 1.8. Let $a_X : X \to \mathrm{pt}$. Define $\omega_X := a_X^! \underline{k}_{\mathrm{pt}}$ and the functor $\mathbf{D} : D^b_c(X, k) \to D^b_c(X, k)$ by

$$\mathbf{D}(F) = R\mathcal{H}\mathrm{om}(\mathcal{F}, \omega_X)$$

Remark. For any $f : X \to Y$ there are natural isomorphisms $f_* \mathbf{D}(\mathcal{F}) = \mathbf{D}(f_! \mathcal{F})$.

2 Perverse sheaves and their basic properties

Definition 2.1. The *perverse t-structure* on X is the t-structure

$${}^p D^b_c(X, k)^{\leq 0} = \{\mathcal{F} \in D^b_c(X, k) : \forall i, \dim \operatorname{supp} \mathcal{H}^i(\mathcal{F}) \leq -i\}$$

$${}^p D^b_c(X, k)^{\geq 0} = \{\mathcal{F} \in D^b_c(X, k) : \forall i, \dim \operatorname{supp} \mathcal{H}^i(\mathbf{D}\mathcal{F}) \leq -i\}$$

Let $\mathrm{Perv}(X, k) = {}^p D^b_c(X, k)^{\leq 0} \cap {}^p D^b_c(X, k)^{\geq 0}$.

Remark. If k were not a field, there would be a subtlety where we'd need the "modified dimension of support" in the latter.

We need to do some work to confirm that this is a t-structure, but once that's done, we automatically get that $\mathrm{Perv}(X, k)$ is abelian, and we have truncation functors ${}^p\tau^{\leq n}, {}^p\tau^{\geq n} : D^b_c(X, k) \to D^b_c(X, k)$ and perverse cohomology sheaf functors ${}^p\mathcal{H}^n : D^b_c(X, k) \to \mathrm{Perv}(X, k)$.

Let $(\mathcal{P}^{\leq 0}, \mathcal{P}^{\geq 0})$ be the perverse t-structure and $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be the standard t-structure.

Lemma 2.2. *Let $j : U \hookrightarrow X$ be an open embedding and $i : Z \hookrightarrow X$ a closed embedding.*

- $\mathcal{F} \in \mathcal{P}^{\leq 0}$ *(resp. $\mathcal{P}^{\geq 0}$)* $\equiv j^* \mathcal{F}, i^* \mathcal{F} \in \mathcal{P}^{\leq 0}$ *(resp. $\mathcal{P}^{\geq 0}$).*

- i_* *preserves* $\mathcal{P}^{\leq 0}$ *and* $\mathcal{P}^{\geq 0}$.
- $i^!$ *and* j_* *preserve* $\mathcal{P}^{\geq 0}$
- $j_!$ *preserves* $\mathcal{P}^{\leq 0}$

So not all preserve perverse sheaves, but this gives us valuable information.

Theorem 2.3. $({}^pD^b_c(X,k)^{\leq 0}, {}^pD^b_c(X,k)^{\geq 0})$ *form a bounded* t*-structure.*

Proof sketch.

- $\mathcal{P}^{\leq -1} := \mathcal{P}^{\leq 0}[1] \subset \mathcal{P}^{\leq 0}$ and $\mathcal{P}^{\geq -1} := \mathcal{P}^{\leq 0}[-1] \supset \mathcal{P}^{\geq 0}$.

  Follows from definitions.

- If $\mathcal{F} \in \mathcal{P}^{\leq -1}$ and $\mathcal{G} \in \mathcal{P}^{\geq 0}$, then $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$.

  Applying natural truncation functors, WLOG $\mathcal{F} = H^j(\mathcal{F})[-j]$. Choose a stratification so that supp $\mathcal{F}$ is a union of strata and $\mathbf{D}\mathcal{G}$ is constructible, and induct on the number of strata.

  Let $i : X_t \hookrightarrow X$ be a closed stratum and $j : X \setminus X_t =: U \hookrightarrow X$. Base case: U is empty. Otherwise, there is an exact sequence

  $$\to \mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(j^*\mathcal{F}, j^*\mathcal{G}) \to$$

  The last term vanishes by induction, so we want to show the first does too. If $i^*\mathcal{F} \neq 0$ (otherwise we're done), then $X_t \subset \dim \mathrm{supp}\, \mathcal{H}^j(\mathcal{F})$, so $\dim X_t \leq -j$. From this, can show $i^!\mathcal{G} \subset \mathcal{T}^{\geq j+1}$. So $\mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) = 0$ by the desired property of $\mathcal{T}$.

- Any $\mathcal{F}$ admits a distinguished triangle $\mathcal{A} \to \mathcal{F} \to \mathcal{B}$ with $\mathcal{A} \in \mathcal{P}^{\leq -1}$ and $B \in \mathcal{P}^{\geq 0}$.

  By noetherian induction, assume this is true for any closed proper subset of $\mathcal{F}$. Let $j_u : X_u \hookrightarrow X \hookleftarrow Z : i$ be an open stratum and its closed complement. Define

  $$\mathcal{G} := \mathrm{Cone}(j_{u!}\tau^{\leq -m-1} j_u^* \mathcal{F} \to j_{u!} j_u^* \mathcal{F} \to \mathcal{F})$$

  and define

  $$\mathcal{B} := \mathrm{Cone}(i_* {}^p\tau^{\leq -1} i^! \mathcal{G} \to i_* i^! \mathcal{G} \to \mathcal{G}) \to \mathcal{G})$$

  Can check that this along with $\mathcal{A} = \mathrm{Cone}(\mathcal{F} \to \mathcal{B})[-1]$ satisfies the condition.

- Bounded: any X is contained in some $\mathcal{P}^{\leq n}$ and some $\mathcal{P}^{\geq n}$.

  In fact, $\mathcal{T}^{\leq -\dim X} \subset \mathcal{P}^{\leq 0} \subset \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0} \subset \mathcal{P}^{\geq 0} \subset \mathcal{T}^{\geq -\dim X}$. Follows from definitions and that Hom vanishing uniquely characterizes the pieces of a t-structure.

□

Remark. Riemann Hilbert correspondence: there is an equivalence from the category of holonomic D-modules on X with regular singularities to the category of perverse sheaves on X.

3 Intermediate extension and the intersection complex

Definition 3.1. Given a locally closed embedding $h :\hookrightarrow X$, the *intermediate extension functor* $h_{!*} : \mathrm{Perv}(Y,k) \to \mathrm{Perv}(X,k)$ is given by

$$h_{!*}(\mathcal{F}) = \mathrm{im}({}^p\mathcal{H}^0(h_!\mathcal{F}) \to {}^p\mathcal{H}^0(h_*\mathcal{F}))$$

The key ingredient in the following results is

$$h_!h^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F}$$

together with properties of t-structures.

Lemma 3.2. *For $h : Y \hookrightarrow X$ a locally closed embedding, $h_{!*} : \mathrm{Perv}(Y,k) \to \mathrm{Perv}(X,k)$ is fully faithful, and $h_{!*}\mathcal{F} \in \mathrm{Perv}(X,k)$ is unique up to isomorphism with the following properties: supported on $\overline{Y}$, $h^*\mathcal{F} \cong \mathcal{F}$, and has no nonzero subobjects or quotients supported on $\overline{Y} \setminus Y$.*

Remark. Let $\overline{Y} \setminus Y =: Z \xrightarrow{i} X$. Then $\mathcal{F}$ having no subobjects/quotients supoprted on Z is equivalent to $i^!\mathcal{F} \in {}^pD^b_c(Z,k)^{\geq 1}$ and $i^*\mathcal{F} \in {}^pD^b_c(Z,k)^{\leq -1}$.

Lemma 3.3. *Let $i : Z \hookrightarrow X$ be a closed subvariety. Then ${}^p\mathcal{H}^0(i_*i^!\mathcal{F})$ is the unique maximal subobject of $\mathcal{F}$ supported on Z, and ${}^p\mathcal{H}^0(i_*i^*\mathcal{F})$ is the unique maximal quotient supported on Z.*

Lemma 3.4. *Let X be an irreducible variety and $j : U \hookrightarrow X \hookleftarrow Z : i$, and let $\mathcal{F} \in \mathrm{Perv}(X,k)$.*

- *If $\mathcal{F}$ has no quotient supported on Z, $0 \to {}^p\mathcal{H}^0(i_*i^!\mathcal{F}) \to \mathcal{F} \to j_{!*}(\mathcal{F}|_U) \to 0$ is exact.*
- *If $\mathcal{F}$ has no subobject supported on Z, $0 \to j_{!*}(\mathcal{F}|_U) \to \mathcal{F} \to {}^p\mathcal{H}^0(i_*i^*\mathcal{F}) \to 0$ is exact.*

Definition 3.5. Let X be a variety, let $h : Y \hookrightarrow X$ be a smooth, connected, locally closed subvariety, and let $\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(X,k)$. The *intersection cohomology complex* associated to $(Y,\mathcal{L})$ is

$$\mathrm{IC}(Y,\mathcal{L}) := h_{!*}(\mathcal{L}[\dim Y]) \in \mathrm{Perv}(X,k)$$

Remark. Taking the hypercohomology of $IC(X;k)[-\dim X]$ yields the *intersection cohomology*, which was actually defined before perverse sheaves.

Example. Let X be a smooth connected variety, $n = \dim X$, let $\mathcal{L} \in \mathrm{Loc}^{\mathrm{ft}}(X,k)$. If $U \subset X$, then $IC(U,\mathcal{L}|_U) \cong \mathcal{L}[n]$.

4 The noetherian and artinian properties

The following results are proved using noetherian induction and the exact sequences for $\mathcal{F}$ subobject/quotient of $j_{!*}(\mathcal{F}|_U)$ with no quotients/subobjects supported on $X \setminus U$.

Lemma 4.1. $\mathrm{Loc}^{\mathrm{ft}}(X,k)[n] \subset \mathrm{Perv}(X,k)$ *is a Serre subcategory.*

Theorem 4.2. *Every perverse sheaf admits a finite filtration whose subquotients are intersection cohomology complexes.*

Lemma 4.3. *Let $Y \subset X$ be smooth, connected, locally closed subvariety. Let $0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{L}'' \to 0$ be a short exact sequence in* $\mathrm{Loc}^{\mathrm{ft}}(Y, k)$. *Then $IC(\mathcal{Y}, \mathcal{L})$ admits a three-step filtration with graded pieces $IC(Y, \mathcal{L}')$, $\mathcal{G}$ supported on $\overline{Y} \setminus Y$, and $IC(Y, \mathcal{L}'')$.*

Theorem 4.4. $\mathrm{Perv}(X, k)$ *is noetherian.*

Proof sketch. Induct on the length of the filtration. Since extension of noetherians is noetherians, suffices to show intersection cohomology complexes are noetherian. By noetherian induction on X, complexes supported on proper closed subsets are noetherian, so suffices to consider $IC(U, \mathcal{L})$ for U open, with $Z = X \setminus U$.

A chain $\mathcal{F}_1 \subset \mathcal{F}_2 \subset ...IC(U, \mathcal{L})$ restricts to a chain of (shifted) local systems on U, which must stabilize to some $\mathcal{L}'$, so by lemma we get $\mathcal{G} \in \mathrm{Perv}(X, k)$ with $IC(U, \mathcal{L}') \subset \mathcal{G} \subset IC(U, \mathcal{L})$.

From the fact that $IC(U, \mathcal{L})$ has no subobject supported on Z, we deduce

$$IC(U, \mathcal{L}') \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset ... \subset \mathcal{G}$$

Taking the quotient of $\mathcal{G}$ by this filtration gives a chain of subobjects of a perverse sheaf supported on Z, so by induction it's eventually constant.

□

The following results require that we work over k a field.

Theorem 4.5.

1. *$IC(Y, \mathcal{L}) \in \mathrm{Perv}(X, k)$ is a simple object for $Y \subset X$ smooth, connected, locally closed subvariety and $\mathcal{L}$ an irreducible local system on Y.*

2. *Every perverse sheaf admits a finite filtration whose subquotients are simple intersection cohomology complexes.*

3. $\mathrm{Perv}(X, k)$ *is artinian, and the simple objects are intersection cohomology complexes.*

Proof sketch.

1. Suppose $0 \neq F \subset IC(Y, \mathcal{L})$. Then $F|_Y \neq 0$ is a sub local system of $\mathcal{L}$, and by irreducibility it is $\mathcal{L}[n]$. Since $\mathcal{F}$ has no non-zero objects supported on $X \setminus Y$, we get an injection $IC(Y, \mathcal{L}) \hookrightarrow F$, and we can conclude it is an equality.

2. Follows from a combination of the two filtration results, and the fact that local systems correspond to $\mathrm{rep}_k^{\mathrm{f.d.}}(\pi_1(Y, y_0))$, which can be broken up into irreducible representations. Thus, we get a filtration for $IC(Y, \mathcal{L})$ whose subquotients are $IC(Y, \mathcal{L}')$ for $\mathcal{L}'$ an irreducible subquotient, or elements of $\mathrm{Perv}(\overline{Y} \setminus Y, k)$ which are covered by noetherian induction..

3. Follows from the previous two.

□