Mixed Hodge structures

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§1. (Polarizable) (Pure) $\mathbb{Q}$-Hodge structures

<u>Def</u> - A (pure) HS of wt k is a f.dim'l $\mathbb{Q}$-vs V w/ a decomp. (contained in datum)

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{pq} \qquad \text{s.t.} \quad \overline{V^{pq}} = V^{qp}.$$

- Given two HS V, W, a hom. $f: V \to W$ is a $\mathbb{Q}$-v hom s.t. $f_{\mathbb{C}}: V^{pq} \longrightarrow W^{pq}$. (In part. wt $V \neq$ wt $W \Rightarrow f=0$)

<u>Def</u> A (pure) HS is a formal direct sum of HS of various weights

<u>Check</u> V, W: HS $\Rightarrow$ $V \oplus W$: HS, $V \otimes W$: HS.

<u>Prime Ex</u> X: sm proj var/$\mathbb{C}$. Then its singular $\mathbb{Q}$-cohomology $H^k(X, \mathbb{Q})$ admits a canonical HS of wt k.
$H^*(X, \mathbb{Q})$ is a HS.

Say V: HS of wt k.

Def The Tate structure is a wt $-2m$ HS
$\mathbb{Q}(m) := (2\pi\sqrt{-1})^m \cdot \mathbb{Q} \subset \mathbb{C}$ w/ only type $(-m,-m)$.
The tate twist of a HS V is
$V(m) := V \otimes \mathbb{Q}(m)$: HS of wt $k-2m$.

Def The Weil operator on V is $C : V_{\mathbb{R}} \longrightarrow V_{\mathbb{R}}$ s.t.
$C : V^{pq} \longrightarrow V^{pq}$ is a mult. by $\sqrt{-1}^{\,q-p}$.
(Check C is def/$\mathbb{R}$ and $C^2 = (-1)^k \cdot \mathrm{id}$)
↳ $C(x+\bar{x}) = Cx + \overline{Cx}$ for $x \in V^{pq}$

Def A polarization on V is a HS hom.
$\varphi : V \otimes_{\mathbb{Q}} V \longrightarrow \mathbb{Q}(-k)$ s.t.
① φ is $(-1)^k$-symmetric
② $V_{\mathbb{R}} \otimes_{\mathbb{R}} V_{\mathbb{R}} \longrightarrow \mathbb{R}$, $x \otimes y \longmapsto (2\pi\sqrt{-1})^k \cdot \varphi(Cx \otimes y)$
is positive defn symm. bilinear form
(Check In fact, ② $\Rightarrow$ ①)

A polarizable HS is a HS w/ at least one polarization

<u>Prime Ex</u> X: sm proj var$/\mathbb{C}$. Then $H^k(X,\mathbb{Q})$ is a wt k polarizable HS. To endow it w/ a polarization, we need to first fix an ample class $\omega \in NS(X)$ and define a Lefschetz triple L, H, Λ in $\mathfrak{gl}(H^*(X,\mathbb{Q}))$.

$\Rightarrow sl_2$-module on H^* $\Rightarrow$ isotypic decomp.

$$\therefore H^k = H^k_{prim} \oplus L \cdot H^{k-2}$$
$$= H^k_{prim} \oplus L \cdot (H^{k-2}_{prim} \oplus L \cdot H^{k-4}) = \cdots\cdots$$

Each H^k_{prim} admits a $(-1)^k$-symmetric HS hom

$$\varphi : H^k_{prim} \otimes H^k_{prim} \longrightarrow \mathbb{Q}(-k)$$
$$x \otimes y \longmapsto (-1)^k \cdot \int_X x \cdot y \cdot \omega^{n-k}$$

The Hodge-Riemann bil. relation thm shows it is a polarization, ie, $(2\pi\sqrt{-1})^k \cdot \int_X (\sqrt{-1}^{\,q-p}) \cdot x \cdot \bar{x} \cdot \omega^{n-k} > 0$

for $\forall\, x \in H^{p,q}_{prim} \subset H^k_{prim,\mathbb{C}}$

The following is an equiv. defn

<u>Def</u> (ver 2) A HS of wt k is a $\mathbb{Q}$-vs V w/ a "decreasing" filtr. $V_{\mathbb{C}} = \cdots \supset F^0 \supset F^1 \supset F^2 \supset \cdots$ s.t.

$$V_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}} \quad \text{for } \forall p.$$

The two defn are related by
$$\begin{cases} V^{pq} = F^p \cap \overline{F^q}, \\ F^p = \bigoplus_{i \geq p} V^{i,q}. \end{cases}$$

<u>Ex</u> X: sm proj$/\mathbb{C}$. Then $H^k(X,\mathbb{Q})$: wt k HS. Its Hodge filtr. is as follows:

de Rham: $\mathbb{C} \overset{q.i.s}{\cong} [\mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n] =: \Omega_X^\bullet$

<u>Thm</u> $\exists$ Hodge-de Rham spectral seq.

$$E_1^{pq} := H^q(X, \Omega_X^p) \Rightarrow E^{p+q} = H^k(X,\mathbb{C}).$$

In fact, it degenerates at 1^{st} page.

Set
$$\begin{array}{c} F^p\Omega_X^\bullet := [\overset{0}{0} \to \cdots \to \overset{p-1}{0} \to \overset{p}{\Omega^p} \to \overset{p+1}{\Omega^{p+1}} \to \cdots \to \overset{n}{\Omega^n}] \\ \downarrow \\ \Omega_X^\bullet = [\mathcal{O} \to \cdots \to \Omega^{p-1} \to \Omega^p \to \Omega^{p+1} \to \cdots \to \Omega^n] \end{array}$$

Then $F^p := \operatorname{im}(H^k(X, F^p\Omega_X^\bullet) \longrightarrow H^k(X,\Omega_X^\bullet) = H^k(X,\mathbb{C}))$.

But note that this doesn't prove $V_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}}$

(we never used the $\mathbb{Q}$-str. on the const. sheaf $\mathbb{C}$.

§2. Mixed Hodge structures

Def A mixed Hodge str. is a f.dim'l $\mathbb{Q}$-vs V equipped w/ two data $W_\bullet$ & $F^\bullet$:

① (weight filtr.) $W_\bullet$ is an increasing filtr.

$$\cdots \subset W_0 \subset W_1 \subset W_2 \subset \cdots = V \qquad (\text{over } \mathbb{Q})$$

② (Hodge filtr.) $F^\bullet$ is a decreasing filtr.

$$V_{\mathbb{C}} = \cdots \supset F^0 \supset F^1 \supset F^2 \supset \cdots \qquad (\text{over } \mathbb{C})$$

③ $Gr_k^W V := W_k / W_{k-1}$ w/ an induced Hodge filtr.

$$Gr_{k,\mathbb{C}} = \cdots \supset F^0 \cap W_{k,\mathbb{C}} / F^0 \cap W_{k-1,\mathbb{C}} \supset \cdots$$ is a pure HS of wt k.

A hom. between two MHS $V \xrightarrow{f} W$ is a $\mathbb{Q}$-vs hom respecting both filtrations.

Def A MHS is (graded) polarizable if $\forall k$-th graded pieces are polarizable PHS.

Thm ① Category of HS is an ab. cat

② Cat of polarizable HS is an ab. subcat. Moreover, it is semisimple (ie, $\forall$ obj is isom to direct sum of simple ones; there's no extensions)

③ Cat of MHS is an ab. cat.

④ Cat of polarizable MHS is an ab. subcat.

Therefore, MHS is a series of extensions of PHS's.

$$\cdots \subset \underbrace{W_0}_{\substack{\text{wt } 0 \\ \text{HS}}} \subset \underbrace{W_1}_{\substack{\text{wt } 1 \\ \text{HS}}} \subset \underbrace{W_2}_{\substack{\text{wt } 2 \\ \text{HS}}} \subset \underbrace{W_3}_{\substack{\text{wt } 3 \\ \text{HS}}} \subset \cdots = V$$

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Thm (Deligne) Every alg. var $X/\mathbb{C}$ of dim n admits a MHS on its cohomology $H^k(X,\mathbb{Q})$. In fact,

$$H^k(-,\mathbb{Q}) : (\mathrm{Var}/\mathbb{C})^{op} \longrightarrow (\mathrm{MHS}) \quad \text{is a functor.}$$

Moreover,

① Assume X is smooth. Then

(i) For $k \le n$, $H^k(X,\mathbb{Q})$ has wt $[k, 2k]$

(ii) For $k \ge n$, $H^k(X,\mathbb{Q})$ has wt $[k, 2n]$

② Assume X: proper. Then

(i) For $k \le n$, $H^k(X, \mathbb{Q})$ has wt $[0, k]$

(ii) For $k \ge n$, $H^k(X, \mathbb{Q})$ has wt $[2k-2n, k]$

<u>Rmk</u> If X: q-proj then the MHS are polarizable. Even if it's not, it's tempting to say this is true, but I cannot find a reference.

<u>Rmk</u> $H^k_c(X, \mathbb{Q})$, $H_k(X, \mathbb{Q})$, $H^{BM}_k(X, \mathbb{Q})$ also have MHS. This is explained by a more general theory of (mixed) Hodge modules and the 6-functor formalism implemented in their cats.

§3. Examples

<u>Ex</u> Say X: sm proj and $j: U \hookrightarrow X$ open subvar.

Then $j^*: H^k(X,\mathbb{Q}) \longrightarrow H^k(U,\mathbb{Q})$ is a MHS hom.

But ① X: sm proj $\Rightarrow$ $H^k(X,\mathbb{Q})$ has pure wt k

② U: sm $\Rightarrow$ $H^k(U,\mathbb{Q})$ has wt $\geq k$.

$\therefore$ $j^*: H^k(X,\mathbb{Q}) \twoheadrightarrow W_k H^k(U,\mathbb{Q}) \subset H^k(U,\mathbb{Q})$.

In fact, it is known that the first map is surjective

<u>Ex</u> Say C: sm proj curve genus g.

$$Z := \{p_1, \cdots, p_n\} \overset{i}{\hookrightarrow} C \overset{j}{\hookleftarrow} U := C \setminus Z.$$

There's a "localization exact sequence"

(from $i_* i^! \mathbb{Q} \to \mathbb{Q} \to Rj_* j^{-1}\mathbb{Q}$)

$$0 \to \cancelto{0}{H^0_Z(C,\mathbb{Q})} \to \overset{1}{H^0(C,\mathbb{Q})} \to \cancelto{0}{H^0(U,\mathbb{Q})}$$

$$\to \cancelto{0}{H^1_Z(C,\mathbb{Q})} \to \overset{2g}{H^1(C,\mathbb{Q})} \to \overset{2g+n-1}{H^1(U,\mathbb{Q})}$$

$$\xrightarrow{\text{Res}} \overset{n}{H^2_Z(C,\mathbb{Q})} \to \overset{1}{H^2(C,\mathbb{Q})} \to \cancelto{0}{H^2(U,\mathbb{Q})} \to 0$$

This is a seq of MHS. It is a top'l fact that

$$H^0(U,\mathbb{Q}) = H^2(U,\mathbb{Q}) = 0, \quad H^1(U,\mathbb{Q}) \cong \mathbb{Q}^{2g+n-1}.$$

It is also a top'l fact that $H^0_Z(C,\mathbb{Q}) = H^1_Z(C,\mathbb{Q}) = 0$ and $H^2_Z(C,\mathbb{Q}) \cong \mathbb{Q}^n$ ($\because\ i^!\mathbb{Q} = \mathbb{Q}[-2](-1)$). In fact, $H^2_Z(C,\mathbb{Q}) = \mathbb{Q}(-1)^n$ is a PHS of wt 2.

Finally, we know the PHS of $H^*(C,\mathbb{Q})$.

Altogether, this determines the MHS of $H^1(U,\mathbb{Q})$: It has $W_1 = H^1(C,\mathbb{Q})$, $W_2/W_1 \cong \ker\left(H^2_Z(C,\mathbb{Q}) \to H^2(C,\mathbb{Q})\right) \cong \mathbb{Q}(-1)^{n-1}$

$\Rightarrow$ Hodge numbers
$$\begin{array}{cc} g & g \\ \multicolumn{2}{c}{n-1} \end{array} \qquad \begin{array}{l} \text{wt } 1 \\ \text{wt } 2 \end{array}.$$

<u>Ex</u> Say $f: X \to S$ is a proj. & surjective morphism between sm proj vars. By generic smoothness, $\exists$ dense Zar. open $U \subset S$

$$\begin{tikzcd} F \arrow[r, hook, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & X_U \arrow[r, hook, "j"] \arrow[d, "f_U"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ \{s\} \arrow[r, hook] & U \arrow[r, hook] & S \end{tikzcd}$$

s.t. f_U: sm proj. Say $s \in U$ and $F = X_s$: fiber

<u>Claim</u> $\operatorname{im}\left(H^k(X,\mathbb{Q}) \to H^k(F,\mathbb{Q})\right) = \operatorname{im}\left(H^k(X_U,\mathbb{Q}) \to H^k(F,\mathbb{Q})\right)$ (i.e., we can ignore the sing. fibers)

Since $j^*: H^k(X,\mathbb{Q}) \xrightarrow{\cong} W_k H^k(X_U,\mathbb{Q})$, the restr. map is $W_k H^k(X_U,\mathbb{Q}) \xrightarrow{i^*} H^k(F,\mathbb{Q})$. But the former has wt $2k$

and the latter has wt $= k$ $\Rightarrow$ its image is the same as the image of $i^*: H^k(X_n, \mathbb{Q}) \longrightarrow H^k(F, \mathbb{Q})$.