$P = W$

" Symmetries of H^*(non-compact algebraic varieties) "

- Compact setting and Hard-Lefschetz.

X : smooth proj. variety / $\mathbb{C}$.

$\dim X = n$

$H \subset \mathbb{P}^N \supset X$ $\qquad D = H \cap X$ generic intersection

$\eta := c_1(\mathcal{O}(D))$

$$\cup \eta^k : H^{n-i}(X, \mathbb{Q}) \longrightarrow H^{n+i}(X, \mathbb{Q})$$

Theorem (Hard Lefschetz)

$\cup \eta^k$ is an isomorphism for all $i = 1, \dots, n$.

Example $X = \mathbb{P}^N$ $\qquad H^*(X, \mathbb{Q}) = \mathbb{Q}[t]/(t^{N+1})$

$$\eta = t.$$

Non-example $X = \mathbb{A}^N \subset \mathbb{P}^N$ $\qquad H^*(X, \mathbb{Q}) = \mathbb{Q}[0]$.

non-compact

- Refinement / relative version

$$X \xrightarrow{f} Y \quad \text{smooth projective}$$

$$R^{n-i} f_* \mathbb{Q}_X \longrightarrow R^{n+i} f_* \mathbb{Q}_X$$

$$Gr_L^{n-i} H^d(X, \mathbb{Q}) \longrightarrow Gr_L^{n+i} H^{d+2i}(X, \mathbb{Q})$$

from Leray filtration

- What happens if

① f not ~~proper~~ projective

② f not smooth ?

If f not smooth, but still projective, then we replace Leray filtration with Perverse filtration.

$$\text{on } D_c(X) \begin{cases} \tau^{\leq} \quad \tau^{\geq} & \text{standard } \tau\text{-struc.} \\ {}^p\tau^{\leq} \quad {}^p\tau^{\geq} & \text{perverse } \tau\text{-struc.} \end{cases}$$

constructible derived category.

given $\mathcal{F} \in D_c$

$$\begin{tikzcd}
{}^p\tau^{\leq k-1}\mathcal{F} \arrow[r] \arrow[dr] & {}^p\tau^{\leq k}\mathcal{F} \arrow[r] \arrow[d] & {}^p\tau^{\leq k+1} \arrow[dl] \\
 & \mathcal{F} &
\end{tikzcd}$$

For $\mathcal{F} = f_* \mathbb{Q}$

$$H^*\left({}^p\tau^{\leq k} f_* \mathbb{Q} \longrightarrow f_* \mathbb{Q} \right) =: P_k H^*(X, \mathbb{Q})$$

derived global sections

Perverse - Hard Lefschetz

$$\mathrm{Gr}^P_{k-1} H^d_{r-i}(X, \mathbb{Q}) \xrightarrow[\cong]{\eta^i \cup} \mathrm{Gr}^P_{r+i} H^{d+2i}(X, \mathbb{Q})$$

Remark : the fact that $\eta^i \cup$ is compatible with the perverse filtration is non trivial.

- <u>Character varieties</u> (twisted)

Σ connected genus g curve.

affine GIT.

$$\mathcal{M}_B(\Sigma, r) = \{ \pi_1(\Sigma, p) \longrightarrow GL_r(\mathbb{C}) \} /\!/ GL_r(\mathbb{C})$$

Character variety

$$\mathcal{M}_B(\Sigma, r, n) = \{ A_1, \dots, A_g, B_1, \dots, B_g \in GL_r(\mathbb{C}) \mid \prod [A_i, B_i] = q_n \} /\!/ GL_r(\mathbb{C})$$

twisted version

$$q_n = e^{2\pi i n / r} \qquad (r, n) = 1$$

Non-compact but smooth whenever

$$\left(H(\mathcal{M}_B(r,n)), W_\bullet, F^\bullet \right)$$ Mixed Hodge structure

Weight filtration

Hodge filtration

<u>Theorem</u> (Mellit, Hausel-Letellier-Rodriguez-Villegas for the conjecture)

$$\exists\, \sigma \text{ s.t. } Gr^W_{2\dim(\mathcal{M})-2i} H^d(\mathcal{M}_B, \mathbb{Q}) \xrightarrow{\cup \sigma^i} Gr^W_{2\dim(\mathcal{M})+2i} H^{d+2i}(\mathcal{M}_B, \mathbb{Q})$$

The Hitchin fibration induces a perverse filtration on the cohomology of $\mathcal{M}_H(r,n)$ + Hard Lefschetz.

Thm (Hitchin, Donaldson, Corlette, Simpson)

$\exists$ C^∞-iso

$$\mathcal{M}_B(r,n) \cong \mathcal{M}_H(r,n)$$

$$\Rightarrow H^*(\mathcal{M}_B(r,n)) \cong H^*(\mathcal{M}_H(r,n))$$

Conjecture (P = W; now a theorem)

$W_{2k} H^*(\mathcal{M}_B(r,n)) \cong P_k H^*(\mathcal{M}_H(r,n))$ by Hausel - Rodriguez-Villegas

<u>Thm</u> P = W is true

Proofs by

- Maulik - Shen
- Hausel - Mellit - Schiffmann
- Maulik - Shen - Yin

this is known as the Courious Hard Lefsetz property.

- Higgs Budles. Σ as above.

semistable Higgs bundles

$$\mathcal{M}_H(r,n) = \{ (E,\vartheta) \quad \vartheta \in \mathrm{Hom}(E, E\otimes\omega) \}^{ss}$$

" some kind of cotagent bundle version of .

$$N(r,n)^{ss} = \{ \text{rank } r \text{ degree } n \text{ vector bulles on } \Sigma \text{, stable} \}$$

$$T_{[E]} N^{s}(r,n) = \mathrm{Ext}^1(E,E)$$

stable

$$\cong \mathrm{Hom}(E, E\otimes\omega)^*$$

$$\dim(\mathcal{M}_H(r,n)) = 2(r^2(g-1)+1)$$

$\mathcal{M}_H(r,n)$ smooth for $(r,n)=1$

$$\mathcal{M}_H(r,n) \longrightarrow \mathbb{A}^{r^2(g-1)+1} = \bigoplus_{i=1}^{r} H^0(\omega^i)$$

$$\vartheta \mapsto \mathrm{charPol}(\vartheta) \in \bigoplus_{i=1}^{r} H^0(\omega^i)$$

$$\mathrm{charPol}(\vartheta) = (\mathrm{tr}\,\vartheta, \mathrm{tr}\,\vartheta^2, \dots, \det\vartheta)$$

Key concepts

$$P = C = W$$

$$\begin{tikzcd} & \mathcal{M}_H(r,d) \times \Sigma \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ \mathcal{M}_H(r,d) & & \Sigma \end{tikzcd}$$

$\mathcal{U}$ universal bundle on $\mathcal{M}_H(r,d) \times \Sigma$

$\gamma_i \in H^i(\Sigma, \mathbb{Q})$ $\quad p_{2*}\left(ch_k(\mathcal{U}) \cup p_1^* \gamma\right)$

$$\| \\ c_k(\gamma) \in H^{i+2k-2}(\mathcal{M}_H(r,d), \mathbb{Q})$$

Theorem (Markman)

$c_k(\gamma)$ generate $H^*(\mathcal{M}_H, \mathbb{Q})$ as an algebra.

Theorem (Shende) $c_k(\gamma)$ has weight $2k$ (seen as a class in $H^*(\mathcal{M}_B, \mathbb{Q})$ via Non-Abelian Hodge isom.

General Fact: $W_\bullet$ is multiplicative w.r.t. $\cup$

Hence $P = W$ reduces to showing analogous properties of $c_k(\gamma)$ with respect to the perverse filtration.

This is not trivial at all though!

Example of NH iso for $r = 1$.

$$\mathcal{M}_B(\overset{1}{r}, n) = (\mathbb{C}^*)^g$$

$$\mathcal{M}_H(\overset{1}{r}, n) = \mathrm{Pic}^n(\Sigma) \times H^0(\omega)$$

$$\cong \mathrm{Pic}^0(\Sigma) \times H^0(\omega)$$

$$\overset{\text{topologically}}{\cong} (S^1)^{2g} \times \mathbb{R}^{2g}$$

Non-abelian Hodge iso is polar coordinates:

$$S^1 \times \mathbb{R} \cong \mathbb{C}^*$$

:)