# Hodge Coniveau of Hypersurfaces

**Def** (Hodge Coniveau) A Hodge structure $H_{\mathbb{C}} = \bigoplus_{p\geq 0} H^{p,k-p}$ has Hodge coniveau $c$ if $H^{p,k-p} = 0$ for $p < c$ and $H^{c,k-c} \neq 0$
($c \leq \frac{k}{2}$) (hence for $p > k-c$)

- If $X$ is a smooth complete intersection in $\mathbb{P}^n$, $\operatorname{codim} X = r$, then by Lefschetz hyperplane theorem $H^k(\mathbb{P}^n) \xrightarrow{\sim} H^k(X)$ for $k < n-r$
$H^{n-r}(\mathbb{P}^n) \hookrightarrow H^{n-r}(X)$.

$\Rightarrow$ Only $H^{n-r}_{prim}(X)$ is interesting.

- By Griffiths theory of Hodge filtration of hypersurfaces (and later generalization by Terasoma, Konno, Dimca, Esnault-Levine-Viehweg ....), we have

**Theorem 1**: $X \subseteq \mathbb{P}^n$ a smooth complete intersection of hypersurfaces of degree $d_1 \leq d_2 \leq \cdots \leq d_r$

Then $H^{n-r}_{prim}(X)$ has Hodge coniveau $\geq c \iff n \geq \sum_{i=1}^{r} d_i + (c-1) d_r$

- **Generalized Bloch conjecture** (for smooth complete intersection on $\mathbb{P}^n$)
(☆) $X \subseteq \mathbb{P}^n$, smooth complete intersection of $r$ hypersurfaces of degree degree $d_1 \leq d_2 \leq \cdots \leq d_r$

If $n \geq \sum_{i=1}^{r} d_i + (c-1) d_r$, then $cl: CH_i(X)_{\mathbb{Q}} \to H^{2n-2r-2i}(X, \mathbb{Q})$ is injective for $i \leq c-1$.

- When $c=1$, $n \geq \sum_{i=1}^{r} d_i \iff X$ is Fano complete intersection

$$\Updownarrow$$

$$H^0(X, K_X) = 0$$

$\Rightarrow X$ is rationally connected $\Rightarrow CH_0(X) = 0$

(So $\circledast$ conjecture holds)

- Unknown for $c \geq 2$

- Theorem (Voisin, 1996)

  For each $(n, d)$, there is $X \subseteq \mathbb{P}^n$ a hypersurface of degree $d$ that satisfies Conjecture $\circledast$.

The theorem depends on the following theorem:

Theorem (Levine – Esnault – Viehweg, 1997)

Let $X \subseteq \mathbb{P}^n$ be a smooth hypersurfaces of degree $d$.

Assume that $X$ is spanned by $r$-planes $\mathbb{P}^r$, i.e. $\forall x \in X$ $\exists$ a linear subspace $\mathbb{P}^r$ of $\mathbb{P}^n$ such that $x \in \mathbb{P}^r \subseteq X$.

Then (1) $CH_\ell(X)_{hom} \otimes \mathbb{Q} = 0$ for $\ell \leq r-1$

(2) $CH_r(X)_{\mathbb{Q}}$ is generated by the classes of $\mathbb{P}^r \subseteq X$.

Proof (of E-L-V)

Let $F = Gr(r, X) \subseteq Gr(r, \mathbb{P}^n)$

↳ parameterizing $\mathbb{P}^r \subseteq X$

$Q = \{(x, P) \in X \times F \mid x \in P\}$ — the universal family

(a) generic finite

$$Q \xrightarrow{q} X \quad \text{(surjective — by assumption)}$$
$$Q \xrightarrow{p} F \quad \text{(projective bundle)}$$

① $q$ is surjective by assumption

② replacing $F$ by its desingularization and its subvariety, we may assume $q$ is generic finite.

(b) Let $h = c_1(\mathcal{O}_X(1))$, $H = q^* h = \mathcal{O}_Q(1)$ as a projective bundle

Claim: for $l < r$, $\cdot H$ is surj.

$$\begin{array}{ccc} & & CH_l(X) \\ & & \downarrow q^* \\ CH_{l+1}(Q) & \xrightarrow{\cdot H} & CH_{l}(Q) \\ \downarrow q_* & & \downarrow q_* \\ CH_{l+1}(X) & \xrightarrow{\cdot h} & CH_l(X) \\ & \searrow \iota_* & \nearrow \iota^* \\ & CH_{l+1}(\mathbb{P}^n) & \end{array}$$

$\iota : X \hookrightarrow \mathbb{P}^n$

$q_* q^* = \deg(q) \cdot \mathrm{Id}$

by projection formula

Commute up to $d$ since $[X] = dh$ in $\mathrm{Pic}(\mathbb{P}^n)$

① $q_* q^* = (\deg q)\,\mathrm{Id}$

② $q_*(q^* h \cdot \alpha) = h \cdot q_* \alpha$ (Projection formula)

③ $l_* l^* = d\,l$

④ $Q$ is a projective bundle of rank $r$ over $F$

So we have a decomposition of $CH(Q)$ compatible with Leray–Hirsch theorem for cohomology:

- $CH_l(Q) = \bigoplus_{0 \le k \le r} H^k p^* CH_{l-r+k}(F)$

- $H^m(Q) = \bigoplus_{0 \le k \le r} [H]^k p^* H^{m-2k}(F)$

$\Rightarrow$ For $l < r$, $CH_{l+1}(Q) \xrightarrow{\cdot H} CH_l(Q)$

and $CH_{r+1}(Q) \xrightarrow{\cdot H} CH_r(Q)$ is surjective except for $p^* CH_0(F)$

⑤ Everything is compatible with class mapp, thus we can consider $CH_l(\ )_{hom}$ version

⑥ $CH_l(\mathbb{P}^n)_{hom} = 0 \implies (1)$ is proved

For (2), $q_* CH_0(F)$ is generated by $\mathbb{P}^r$ planes. $\square$

Lemma (Kollár)

$X = V(f) \subseteq \mathbb{P}^n$, $f$ = degree $d$ with $d \leq n$

Then $\mathbb{P}^n$ is covered by lines $l$ with either $l \cap X$ = point
or $l \subseteq X$.

(Pf) Say $x = 0 \in \mathbb{A}^n \subseteq \mathbb{P}^n$ ($\in V(f)$), $\lambda \in \mathbb{P}^{n-1}$

$l = \{ t\lambda \mid t \in \mathbb{A}^1 \}$

Then $f(t\lambda) = t^d f_d(\lambda) + \cdots + f_0(\lambda)$, $f_0(\lambda) = 0$ since $x \in V(f)$.

So $l \cap X = \{x\}$ $\iff$ $f_{d-1}(\lambda) = \cdots = f_1(\lambda) = 0$
or $l \subseteq X$

$\{x\}$ ← $f_d(\lambda) \neq 0$

$l \subseteq X$ ← $f_d(\lambda) = 0$

(Only $d-1$ equations. Since $d \leq n$, so $\forall x$, there is some $l$ with $l \cap X = \{x\}$ or $l \subseteq X$.) □

Voisin's examples : Write $n = cd + s$

Let $f_1 \in \mathbb{C}[X_0, \ldots, X_d]$

$f_2, \ldots, f_{c-1} \in \mathbb{C}[X_1, \ldots, X_d]$

$f_c \in \mathbb{C}[X_1, \ldots, X_{d+s}]$

Set $f(X_0, \ldots, X_{cd+s}) = f_1(X_0, \ldots, X_d) + f_2(X_{d+1}, \ldots, X_{2d}) + \cdots$

$\cdots + f_{c-1}(X_{(c-2)d+1}, \ldots, X_{(c-1)d}) + f_c(X_{(c-1)d+1}, \ldots, X_{cd+s})$

Claim : $\forall x \in X = V(f)$, there exists $\mathbb{P}^{c-1}$ such that $\mathbb{P}^{c-1} \cap X = \mathbb{P}^{c-2}$ ($x \in$)

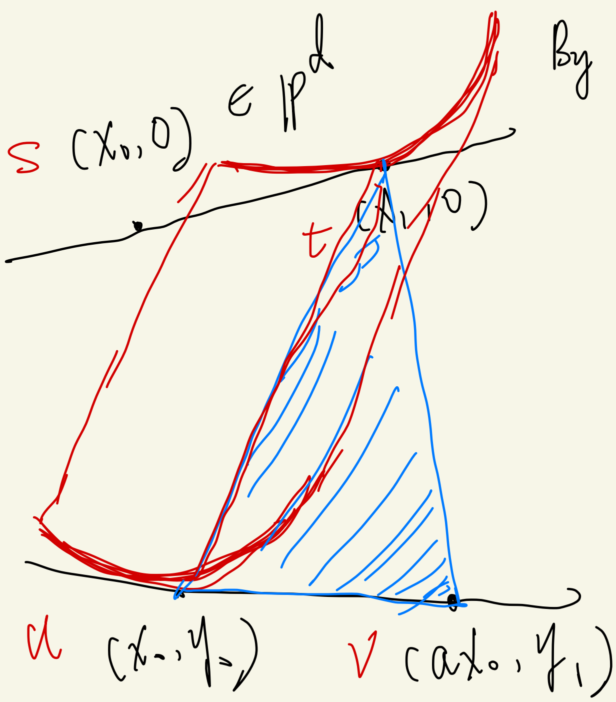
$\cdot \in \mathbb{P}^{c-1} \subseteq X$

For simplicity, consider the case $c=2$, $n=2d+s$, $s\geq 0$

$$f = f_1(X_0,\dots,X_n) + f_2(Y_1,\dots,Y_{d+s}), \quad \mathbb{P}^{2d+s}$$

$$\mathbb{P}^d \subseteq \mathbb{P}^{2d+s} \supseteq \mathbb{P}^{d+s-1}$$
$$[x,0] \qquad [0,y]$$

- Given $(x_0,y_0) \in \mathbb{P}^{2d+s}$, may assume $x_0, y_0 \neq 0$.




By lemma, $\exists\, \Delta \subseteq \mathbb{P}^d$ s.t. $f(sx_0+tx_1) = s^d$
$[s,t]$

$$= f_1(sx_0+tx_1)$$

$$\langle [x_0,0], \mathbb{P}^{d+s-1} \rangle = \mathbb{P}^{d+s}$$

By lemma, $\exists\, \Delta' \subseteq \mathbb{P}^{d+s}$ s.t. $f(u(x_0,y_0)+v(ax_0+y_1)) = v^d$

$$f_1((u+av)x_0) + f_2(uy_0+vy_1)$$
$$(u+av)^d$$

Consider $\mathbb{P}^2 = \langle [x_0,0], \Delta' \rangle$.

Then $f(t[x_1,0]+u[x_0,y_0]+v[ax_0,y_1]) = (u+av)^d + v^d - (u+av)^d$
$= \underline{v^d}$

So $\mathbb{P}^1 = \langle [X_*, 0], [X_0, Y_0] \rangle = X \cap \mathbb{P}^2$ 1

(In general, prove by induction.) □

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## Griffiths' theory

$X$ : smooth proj. variety, $\dim X = n$, $U = X \setminus Y \xhookrightarrow{j} X$

$\cup\!|$

$Y$ : smooth hypersurface $Y \xrightarrow{\iota} X$

**Def** (Pole Order Filtration)

On the complex $\Omega_X^\bullet(*Y)$, where $\Omega_X^l(*Y) := \varinjlim_k \Omega_X^l(kY)$,

we define

$$P^p\Omega_X^\bullet(*Y) : 0 \to \cdots \to 0 \to \Omega_X^p(Y) \to \Omega_X^{p+1}(2Y) \to$$

(Deligne, inspired by Griffiths) $\cdots \to \Omega_X^d((d-p+1)Y) \to \cdots$

**Theorem**: There are filtered quasi-isomorphisms

"P = F"

$$(\Omega_X^\bullet(\log Y), \tau) \xrightarrow{\text{q.is.}} (\Omega_X^\bullet(*Y), P) \xrightarrow{\text{q.is.}} (j_*\Omega_U^\bullet, \tau)$$

↑ bête filtration

In particular, $P^pH^k(U,\mathbb{C}) = F^pH^k(U,\mathbb{C})$

**Remark**: If $X = \mathbb{P}^n$ and $Y$ is a singular hypersurface, then Deligne–Dimca shows that $P^pH^k(U,\mathbb{C}) \supseteq F^pH^k(U,\mathbb{C})$, and are not equal in general.

## Assumption

(∗) $H^i(X, \Omega_X^p(kY)) = 0$ for $k \geq 1$, $i \geq 1$, $p \geq 0$

**Remark**: (∗) condition holds if

(a) $Y$ is sufficiently ample

or (b) $Y$ is only ample but $X$ satisfies Bott's vanishing theorem (Eg. $X = \mathbb{P}^n$ or $X$ is toric)

**Corollary** (Griffiths)

- Under (∗) assumption, $P^p\Omega_X^\bullet(*Y)$ is $\Gamma$-acyclic.

$$\Rightarrow F^p H^d(U, \mathbb{C}) = \mathbb{H}^d(X, P^p\Omega_X^\bullet(*Y)) = H^d(\Gamma(X, P^p\Omega_X^\bullet(*Y)))$$

In particular, for $d = n$, $F^pH^n(U, \mathbb{C}) = \dfrac{H^0(\Omega_X^n((d-p+1)Y))}{d\,H^0(\Omega_X^{n-1}((d-p)Y))}$

- On the other hand, there is an exact sequence of MHS:

$$H^k(X) \to H^k(U) \to H^{k+1}(X, U) \to H^{k+1}(X) \to H^{k+1}(U)$$

$H^{k+1}(X, U)$ $\cong$ (excision) $H^{k+1}(N_{Y/X}, N_{Y/X} - Y)$ $\cong$ (Thom's iso.) $H^{k-1}(Y)(1)$

$\frac{1}{2\pi i}\mathrm{Res}: H^k(U) \to H^{k-1}(Y)(1)$

$\iota_*: H^{k-1}(Y)(1) \to H^{k+1}(X)$ (Gysin map) (Poincaré dual of push forward of homology)

- Now assume $Y$ is ample and $k=n$. Then we have SES

$$0 \to H^n_{prim}(X) \xrightarrow{j^*} H^n(U) \xrightarrow{\frac{1}{2\pi i}\mathrm{Res}} H^{n-1}_{van}(Y)(-1) \to 0$$

$$H^n_{prim}(X) = \frac{H^n(X)}{LH^{n-2}(X)} = \frac{H^n(X)}{\iota_* H^{n-2}(Y)} = W_n H^n(U)$$

$$H^{n-1}_{van}(Y) = \ker\left(H^{n-1}(Y) \xrightarrow{\iota_*} H^{n+1}(X)\right) = Gr_W^{n+1} H^n(U)$$

- From now on, let $X = \mathbb{P}^n$ and $Y$ = degree $d$ hypersurface.

$$\Rightarrow \quad F^{n-p+1} H^n(U, \mathbb{C}) \xrightarrow[\sim]{\frac{1}{2\pi i}\mathrm{Res}} F^{n-p} H^{n-1}_{prim}(Y, \mathbb{C})$$

$$F^{n-p+1} H^n(U,\mathbb{C}) = \frac{H^0(\mathbb{P}^n, K_{\mathbb{P}^n}(pY))}{d H^0(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}((p-1)Y))}$$

(mod $F^{n-p+1}$)

$$\to H^{n-p,p-1}_{prim}(Y, \mathbb{C})$$

$$\frac{H^0(\mathbb{P}^n, K_{\mathbb{P}^n}(pY))}{d H^0(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}((p-1)Y)) + H^0(\mathbb{P}^n, K_{\mathbb{P}^n}((p-1)Y))} \xrightarrow{\sim} H^{n-p,p-1}_{prim}(Y, \mathbb{C})$$

**Theorem** (Griffiths)

If we identify $H^0(K_{\mathbb{P}^n}(pY)) \simeq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(pd-n-1)) = S^{pd-n-1}$,

then $dH^0(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}((p-1)Y)) + H^0(\mathbb{P}^n, K_{\mathbb{P}^n}((p-1)Y)) \simeq J_f^{pd-n-1}$

In particular, $R_f^{pd-n-1} \simeq H^{n-p,p-1}_{prim}(Y)$

$S = \mathbb{C}[x_0, \dots, x_n]$, $Y = V(f)$

$J_f := \langle \frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_i} \rangle$

$R_f := \frac{S}{J_f}$

**Remark**: We know exactly when $R_f^k \neq 0$.

**Theorem** (Macaulay)

Let $G_i$ $(i = 0, \dots, n)$ be a regular seq. of homogeneous polynomials of degree $d_i$ (in $S$). Set $R_G = \frac{\mathbb{C}[x_0, \dots, x_n]}{\langle G_0, \dots, G_n \rangle}$.

Then $R_G$ is graded Gorenstein with socle in degree $N$.

$N = \sum_{i=0}^{n} d_i - n - 1$

(In other words, there is a perfect pairing $R_G^k \otimes R_G^{N-k} \to R_G^N$)

(In particular, $R_G^k \neq 0 \iff 0 \leq k \leq N$)

**Cor**: $H^{n-1}_{prim}(Y)$ has coniveau $\geq c \iff n \geq cd$

(pf) $H^{n-p,p-1}_{prim}(Y) = R_f^{pd-n-1} \neq 0 \iff 0 \leq pd-n-1 \leq (n+1)(d-2)$

always holds for $d \geq 2$, $p \leq \lfloor \frac{n-1}{2} \rfloor$

□

**Complete intersection case**:

**Terasoma's trick**:

$W$ = smooth proj. variety of dim $n$

$\cup$

$X$ = complete intersection of sections of ample line bundles $L_1, \dots, L_r$ (codim $r$)

Set $\mathcal{E} = L_1 \oplus L_2 \oplus \cdots \oplus L_r$ and $X_{\mathcal{E}} = V(\underbrace{f_1, f_2, \dots, f_r}_{\in\, \mathcal{O}_{W_{\mathcal{E}}}(1)}) \subseteq \mathbb{P}(\mathcal{E}) = W_{\mathcal{E}}$

**Observation**:

① $$\begin{array}{ccccc} & & X_{\mathcal{E}} & & \\ & {\scriptstyle k}\nearrow & & \searrow{\scriptstyle j_{\mathcal{E}}} & \\ \mathbb{P}(\mathcal{E}|_X) & & \subseteq & & W_{\mathcal{E}} \\ \downarrow{\scriptstyle \pi_0} & & & & \downarrow \\ X & & \overset{j}{\subseteq} & & W \end{array}$$

and

$$\begin{array}{c} W_{\mathcal{E}} \setminus X_{\mathcal{E}} \\ \downarrow \\ W \setminus X \end{array}$$ is an affine vector bundle

② $(W_{\mathcal{E}}, X_{\mathcal{E}})$ satisfies ⊛ condition if $W$ satisfies

Bott's vanishing.

③ · **Proposition** : $H^{n-r}_{van}(X, \mathbb{Q}) \xrightarrow{i_* \circ \pi_0^*} H^{n+r-2}_{van}(X_Z, \mathbb{Q})$ is an isomorphism of HS of type $(r-1, r-1)$.

(pf) We have

$$
\begin{array}{ccccccccc}
0 \to & \dfrac{H^{n+r-1}(W)}{\mathrm{Im}\, j_*} & \to & H^{n+r-1}(W-X) & \xrightarrow{Res} & H^{n-r}_{van}(X) & \to & 0 \\
 & \downarrow \text{(Claim)} & & \downarrow \wr & & \downarrow i_* \pi_0^* & & \\
0 \to & \dfrac{H^{n+r-1}(W_Z)}{\mathrm{Im}\, j_{Z*}} & \to & H^{n+r-1}(W_Z - X_Z) & \to & H^{n+r-2}_{van}(X_Z) & \to & 0
\end{array}
$$

Claim: $\wr$

- Let $l = c_1(\mathcal{O}_W(1))$. Then by Lefschetz hyperplane theorem

$$\mathrm{Im}\left(H^{n-r-1}(X) \xrightarrow{j_*} H^{n+r-1}(W)\right) = \mathrm{Im}\left(H^{n-r-1}(W) \xrightarrow{c_r(X)} H^{n+r-1}(W)\right)$$

- Also, $H^{n+r-1}(W_Z) = \bigoplus_{i \le r-1} l^i \cup H^{n-r-1-2i}(W)$ and

$$\mathrm{Im}\, j_{Z*} \text{ “=” } \mathrm{Im}(\cup c_1(W)) \text{ “=” } \mathrm{Im}(\cup l)$$

So $\dfrac{H^{n+r-1}(W_Z)}{\mathrm{Im}\, j_{Z*}} = \dfrac{H^{n+r-1}(W)}{\mathrm{Im}(\cup c_r(X))}$ since $l^r = -\sum_{i \le r-1} (-1)^{r-i} l^i \cup \pi_0^* c_{r-i}(E)$ $= 0$

∎

Now for $W = \mathbb{P}^n$, $L_i \cong \mathcal{O}(d_i)$, we have $F^{n-p} H^{n-r}_{prim}(X)$

$\|$

$$H^0(W_E, K_{W_E} \otimes \mathcal{O}_{W_E}(p)) \twoheadrightarrow F^{n+r-p} H^{n+r-1}(U, \mathbb{C}) \twoheadrightarrow F^{n+r-p-1} H^{n+r-2}_{van}(X_E)$$

$\|$

Note: $K_{W_E} \cong \mathcal{O}_{W_E}(-r) + \pi^* \det(\mathcal{E}) + \pi^* K_W$

(by relative Euler seq)

$$H^0(W_E, \mathcal{O}_{W_E}(p-r) \otimes \pi^* \mathcal{O}(\sum_i d_i - n - 1))$$

$\|$ proj. formula

$$H^0(W, S^{p-r}(\oplus \mathcal{O}(d_i)) \otimes \mathcal{O}(\sum d_i - n - 1))$$

So $\_\_\_ = 0$ if $\sum d_i + (p-r)\sup\{d_i\} - n - 1 < 0$

$\square$