LAST TIME: we saw also results about the implication generalized Hodge conj $\Longrightarrow$ generalized Bloch conj.

TODAY: we will pick a different route w/ the same destination and in the meantime

we will describe the work of Kimura on ⊛ Nilpotence conjecture

on ⊛ "finite dimensionality" of Chow rings.

Setting: X smooth proj connected$/\mathbb{C}$. $\Gamma \in CH^k(X\times X)$ a correspondence. Recall that we can compose correspondences:

$$CH^k(X\times X) \times CH^\ell(X\times X) \xrightarrow{\circ} CH^{k+\ell-\dim(X)}(X\times X)$$

$$(\Gamma, \Gamma') \longmapsto \Gamma'\circ\Gamma = p_{13*}(p_{12}^*\Gamma \cdot p_{23}^*\Gamma')$$

⊛ This $\circ$ is associative but not commutative.

⊛ Given $\Gamma \in CH^k(X\times Y)$, we have $\Gamma_*: CH^i(X) \longrightarrow CH^{i+k-\dim(X)}(Y)$, $z \mapsto pr_{2*}(pr_1^*(z)\cdot\Gamma)$

Recall that proj. formula $\Rightarrow$ $(\Gamma'\circ\Gamma)_* = \Gamma'_*\circ\Gamma_*$ (prop. 2.10)

The starting point of Kimura's work is the following (still open) conj:

Nilpotence Conj: let Γ be a correspondence in $CH^*(X\times X)$ such that $\Gamma \underset{\text{hom.}}{\sim} 0$. $\left(CH^*(X\times X)_{\mathbb{Q}} \xrightarrow{cl} H_B^{2*}(X\times X) \right)$

$\Gamma \longmapsto cl(\Gamma)=0$.

Then $\exists N>0$: $\Gamma^{\circ N} = 0$ in $CH^*(X\times X)_{\mathbb{Q}}$.

Rmk: if $\Gamma \in CH^k(X\times X) \Rightarrow \Gamma^{\circ 2} \in CH^{2k-\dim(X)}(X\times X)$:

$k>\dim(X) \Rightarrow k+(k-\dim(X)) = 2k-\dim(X) > k \ (>\dim(X)) \Rightarrow \Gamma^{\circ N} \in CH^{>2\dim(X)}(X\times X)$ for $N\to\infty$

$k=\dim(X) \Rightarrow 2k-\dim(X) = k \ (=\dim(X)) \Rightarrow \Gamma^{\circ N} \in CH^{\dim(X)}(X\times X)$ $\forall N\geq 1$.

$k<\dim(X) \Rightarrow k+(k-\dim(X)) = 2k-\dim(X) < k \ (<\dim(X)) \Rightarrow \Gamma^{\circ N} \in CH^{<0}(X\times X)$ for $N\to\infty$

$\Rightarrow$ The Nilpotence is trivially true for $k\neq\dim(X)$. So from now on we consider $k=\dim(X)$

Recall: $\Gamma \in CH^{\dim(X)}(X \times X)$ is a 0-correspondence: (in particular $\Gamma_*: CH^*(X) \longrightarrow CH^*(X)$ preserves degree)

① Theorem: (Voevodsky '95, Voisin '94) if $\Gamma \sim_{alg} 0 \Rightarrow \exists N >> 0: \Gamma^{\circ N} = 0$ in $CH(X \times X)$.

proof:

Recall: $Z \in \mathcal{Z}^k(Y)$ is alg. eq. to zero if

1) $\exists$ a smooth curve C

2) $z \in \mathcal{Z}_0(C): z \sim_{hom.} 0$

3) $\tilde{\Gamma} \in \mathcal{Z}^k(C \times Y)$

s.t. $Z = \tilde{\Gamma}_*(z)$

$\Rightarrow$ applied to our case: let $z \in CH_0(C): z \sim_{hom.} 0$ and $\Gamma = \tilde{\Gamma}_*(z)$ w/ $\tilde{\Gamma} \in CH^{\dim(X)}(C \times X \times X)$

$$\Gamma^{\circ N} = \underbrace{\tilde{\Gamma}_*(z) \circ \dots \circ \tilde{\Gamma}_*(z)}_{N\text{-times}}$$

Strategy: 1) Notice that $\forall N > 0$ there is a map $CH_0(C) \times \dots \times CH_0(C) \longrightarrow CH^{\dim(X)}(X \times X)$

$$(z_1, \dots, z_N) \longmapsto \tilde{\Gamma}_*(z_1) \circ \dots \circ \tilde{\Gamma}_*(z_N)$$

Idea: define $\forall N > 0$ $\tilde{\Gamma}_N \in CH^{*}(C^N \times X \times X): \tilde{\Gamma}_{N*}(z_1 \times \dots \times z_N) =$ $\Uparrow$ recovers what we had already

where $z_1 \times \dots \times z_N := pr_1^* z \cdot \dots \cdot pr_N^* z$ on C^N.

2) Use that for curves, $\forall$ 0-cycle $z \in CH_0(C)$ if $N \geq g(C)+1$ then $z^N = 0$ in $CH_0(C^N)$.

$\uparrow$ [Voisin]

Rmk:

$* =$ we need to go from N to $N + * - N = \dim(X) \Rightarrow * = \dim(X)$.

Consider $\tilde{\Gamma}^N = \underbrace{\tilde{\Gamma} \times \dots \times \tilde{\Gamma}}_{N\text{-times}} \in CH^*(C^N \times (X \times X)^N)$

Consider $\tilde{\Delta} := \{(x_1, \dots, x_{2N}) \in (X \times X)^N : x_2 = x_3, x_4 = x_5, \dots, x_{2N-2} = x_{2N-1}\}$

Define: $\tilde{\Gamma}_N := pr_{C^N \times X \times X}\left(\tilde{\Gamma}^N \cdot pr_{(X \times X)^N}(\tilde{\Delta})\right)$ where $pr_{C^N \times X \times X}$ projects on x_1 & x_{2N}.

One can check that $\tilde{\Gamma}_{N*}(t_1, \dots, t_N) = \tilde{\Gamma}_*(t_1) \circ \dots \circ \tilde{\Gamma}_*(t_N)$. #

(2) The second big result of Kimura is the following:

Thm: If X is Finite-Dimensional then the Nilpotence Conj. is true for X

§ Finite dimensionality of Chow groups

Let's see what Kimura means by finite-dimensional & why we have such definition.

Evidence #1: C curve, then $CH_0(C)_{deg-0} = J_C(k)$, namely it is parametrized by a fin. dimensional variety.

Evidence #2: However, X higher dim. by thm of Mumford, if $p_g(X) > 0$ (e.g. $X = C \times D$)

$$\Rightarrow CH_0(X)^{deg-0}_{\mathbb{Q}} \text{ is } \infty\text{-dimensional in the following sense:} \quad (h^{\dim(X),0} > 0)$$

$\forall n \geq 1$ the map $\mathrm{Sym}^n S \times \mathrm{Sym}^n S \xrightarrow{\sigma_n} CH_0(S)_{deg-0}$ $(z_1, z_2) \to z_1 - z_2$ is NEVER SURJECT.

$\Rightarrow$ So this def. of fin dimentionality is too strict.

Kimura's idea: let $\alpha_1 \dots \alpha_n \in CH^*(X)_{\mathbb{Q}}$: then define:

alternating prod. $\alpha_1 \wedge \dots \wedge \alpha_n = \sum_{\sigma \in S_n} \frac{\mathrm{sgn}(\sigma)}{n!} \alpha_{\sigma(1)} \times \dots \times \alpha_{\sigma(n)} \in CH^*(X^n)$.

symmetric prod $\alpha_1 \bullet \dots \bullet \alpha_n = \sum_{\sigma \in S_n} \frac{1}{n!} \alpha_{\sigma(1)} \times \dots \times \alpha_{\sigma(n)} \in CH^*(X^n)$.

Kimura's thm: (1) let $S = C \times D$ & $\alpha_1 \dots \alpha_n \in \mathrm{Ker}(CH_0(S)_{deg\,0} \longrightarrow \mathrm{Alb}_S(k))$ given by sending $s \mapsto s - s_0$

If $n > 4p_g(S) \Rightarrow \alpha_1 \wedge \dots \wedge \alpha_n = 0$

(2) But for curves: $\forall n \; \exists \; \alpha_1 \dots \alpha_n \in J_C(k)$: $\alpha_1 \wedge \dots \wedge \alpha_n \neq 0$.

However if $n > 2 \cdot g(C) \Rightarrow \alpha_1 \bullet \dots \bullet \alpha_n = 0$

"DEFINITION": X is said to be finite dimensional if $\exists\, n$ & a decomposition

$$CH^*(X)_{\mathbb{Q}} = CH^*(X)_+ \oplus CH^*(X)_- \text{ s.t. } \forall\, \alpha_1 \ldots \alpha_n \in CH^*(X)_{even} \Rightarrow \alpha_1 \wedge \ldots \wedge \alpha_n = 0 \qquad \Lambda^n CH^*(X)_+ = 0$$

$$\forall\, \alpha_1 \ldots \alpha_n \in CH^*(X)_{odd} \Rightarrow \alpha_1 \cdot \ldots \cdot \alpha_n = 0 \qquad Sym^n CH^*(X)_- = 0$$

DEFINITION': X is said to be finite dimensional if $\exists\, n$ and $\exists$ projectors e_+ & $e_- \in CH^{\dim(X)}(X \times X)_{\mathbb{Q}}$

$[e_+ e_- = e_- e_+ = 0,\ e_\pm^{\circ 2} = e_\pm,\ e_+ + e_- = \Delta_X]$ such that

$\forall\, \alpha_1 \ldots \alpha_n \in CH^*((X, e_+, 0) = X^+) \Rightarrow \alpha_1 \wedge \ldots \wedge \alpha_n = 0 \qquad (\Lambda^n CH^*(X^+) = 0)$

" " " " $(X, e_-, 0) = X^- \Rightarrow \alpha_1 \cdot \ldots \cdot \alpha_n = 0 \qquad (Sym^n CH^*(X^-) = 0)$

Recall: Chow group of a motive $(X, p, 0)$ is: $CH^i(X, p, 0) = p_*(CH^i(X)_{\mathbb{Q}})$

In particular $\Delta_X = e_+ + e_- \Rightarrow CH^*(X)_{\mathbb{Q}} = CH^*(X^+) \oplus CH^*(X^-)$ (so we recover the previous def.)

$$CH^*(X^+) = e_{+*} CH^*(X), \qquad CH^*(X^-) = e_- CH^*(X)$$

Rmk: Consider $\sigma \in S_n$ & $\sigma : X^n \longrightarrow X^n$, $(x_1 \ldots x_n) \longmapsto (x_{\sigma(1)}, \ldots x_{\sigma(n)})$. Call Γ_σ the graph.

Define $\Gamma_+ = \sum_{\sigma \in S_n} \frac{1}{n!} \Gamma_\sigma$ & $\Gamma_- = \sum_{\sigma \in S_n} \frac{sgn(\sigma)}{n!} \Gamma_\sigma \subseteq X^n \times X^n$

Then $\Gamma_{+*}(\alpha_1 \times \ldots \times \alpha_n) = pr_{2*}(\alpha_1 \times \ldots \times \alpha_n \times X^n \cdot \sum_{\sigma \in S_n} \frac{1}{n!} \Gamma_\sigma) = \alpha_1 \cdot \ldots \cdot \alpha_n$

$\Gamma_{-*}(\alpha_1 \times \ldots \times \alpha_n) = \alpha_1 \wedge \ldots \wedge \alpha_n$

$\Rightarrow$ What we ask is that $\Gamma_{-*} \circ (e_{+*}(\sim))^n$ & $\Gamma_{+*} \circ (e_{-*}(\sim))^n$ are both trivial, for some n

$$\Gamma_{-*} \circ (e_{+*}(\sim))^n = \Gamma_{-*} \circ e^n_{+*}(\sim), \qquad \Gamma_{+*} \circ (e_{-*}(\sim))^n = \text{same}.$$

Theorem: product of curves are finite dimensional.

proof: pick a curve C & fix a point $p \in C(\mathbb{C})$. Then:

$e_+ := [p \times C] + [C \times p]$ is a projector $\Rightarrow \Delta_X = \underbrace{\Delta_X - e_+}_{=: e_-} + e_+$

Both need Motive theory.

$\boxed{\text{Thm}}$ $\Gamma_+ \circ e_-^n = 0$ if $n = 2g(C)+1$ and $\Gamma_- \circ e_+^n = 0$ if $n \geq 2g(C)+1$ (larger than the dim of $J \times J$)

$\boxed{\text{Lemma}}$ Let X & Y be finite dimensional $\Rightarrow$ $X \times Y$ is finite dimensional. #

Rmk: actually for surfaces enough to have $C \times D \dashrightarrow S$

Conjecture: any X is finite dimensional. (Kimura & O'Sullivan)

THEOREM (Kimura). Let X be finite dimensional. Let $f \in CH^{\dim(X)}(X \times X)_{\mathbb{Q}}$: then

α satisfies a $\mathbb{Q}$ - polynomial equation : $f^{\circ N} = q_{N-1} f^{\circ N-1} + \dots + q_0 \Delta_X$.

Moreover $q_i = 0$ if $f \sim_{hom} 0$: $\Rightarrow$ Nilpotence conjecture.

Sketch of

proof: $f \in CH^{\dim(X)}(X \times X)_{\mathbb{Q}}$: $f = f_+ + f_- = e_{+*} f + e_{-*} f$

Therefore you can assume that you are dealing $(X, e_+, 0)$ & w/ $(X, e_-, 0)$ which are respectively

Fictive case: $e_+ = \Delta_X$ & $e_- = 0$ & $n = 2$ & f corresponds to a map $f: X \longrightarrow X$ w/ finitely

many fixed points.

The assumption on $e_+ = \Delta_X$ & $n=2$ $\Rightarrow$ $(\Gamma_- \circ \Delta_X^2)_* = 0$ $\Rightarrow$ $\Gamma_{-*} = (\Gamma_{id} - \Gamma_{inv})_* = 0$ (we are multiplying by 2!)

$\Delta_{X \times X}$

$(\Delta_{X \times X} - \Gamma_{inv})_* = 0$ when $\Delta_{X \times X} = \{(x,y,x,y) : x,y \in X\}$; $\Gamma_{inv} = \{(x,y,y,x) : x,y \in X\}$

$\Gamma_- \circ \Gamma_{(f,f)} = 0 \Rightarrow (\Gamma_{id} - \Gamma_{inv}) \circ \Gamma_{(f,f)} = \Gamma_1 - \Gamma_2$

where $\Gamma_1 = \{(x, y, f(x), f(y)) : x, y \in X\}$ and $\Gamma_2 = \{(x, y, f(y), f(x)) : x, y \in X\}$

$\Rightarrow \quad 0 = (\Gamma_1 - \Gamma_2) \cdot p_{13}^*(\Delta_X) = \Gamma_1' - \Gamma_2'$ where $\Gamma_1' : x = f(x)$ - fixedpnt $\Rightarrow \{(x, y, x, f(y)) : x = f(x)\}$

$\Gamma_2' : x = f(y) \Rightarrow \{f(y), y, f(y), f(f(y))\}$

$\Rightarrow 0 = p_{2a\,*}(\Gamma_1' - \Gamma_2') = \#\text{fixedpoint } f \cdot \Gamma_f - \Gamma_{f\circ f} = \deg(\Delta_X \cdot \Gamma_f) \cdot \Gamma_f - \Gamma_f^{\circ 2} = 0$

§ Relations between Nilpotence conj ; Bloch conj & generalized Hodge conj.

THM (Kimura 2005):

Nilpotence Conj $\Rightarrow$ Bloch conj for surfaces : $p_g = q = 0$

Corollary: S surface : $p_g = q = 0$ & S rationally dominated by curves. $\Rightarrow$ Bloch conj holds for S.

proof:

RECALL Bloch Conj: $H^{p,q}(S) = 0$ for $p \neq q$ & $p < 1$ (or $q < 1$). then

$cl : CH_0(S)_{\mathbb{Q}} \overset{inj}{\hookrightarrow} H^4_B(X, \mathbb{Q})$ for $i < 1$

or equiv. $CH_0(S)_{\mathbb{Q}\ hom} = 0$

Lefschetz Theorem on 1,1-form $\Rightarrow$ cl map surjective on $H^2_B(S, \mathbb{Q}) \cap H^{1,1}(S)$

$p_g = 0 \Rightarrow H^2_B(S, \mathbb{Q}) = H^{1,1}(S) = \langle [C_i] \rangle$

$q = 0 \Rightarrow cl([\Delta]) \in H^4_B(X \times X, \mathbb{Q}) \overset{\text{Kunneth}}{=} H^0_B(X) \otimes H^4 \oplus H^2(X) \otimes H^2(X) \oplus H^4 \otimes H^0$

$[\Delta_X] = [X \times \{x\}] + \sum n_{ij} [C_i \times C_j] + [\{x\} \times X]$

$\Gamma = \Delta_X - X \times \{x\} - \sum n_{ij}\, C_i \times C_j - \{x\} \times X \qquad \Gamma \sim_{hom} 0 \Rightarrow \Gamma^{\circ N} = 0$

$\Rightarrow \Gamma^{\circ N}_* : CH_0(X)_{\mathbb{Q}} \longrightarrow CH_0(X)_{\mathbb{Q}}$ is zero ; but but on $CH_0(X)_{hom}$

$X \times \{x\}_* (z) = \deg(z) x = 0$ & $C_i \times C_j{}_* (z) = pr_{2*}(C_i \times C_j \cdot z \times X) = 0$

avoid the curve

But $\Delta_{X*}(z) = z \Rightarrow z = 0$

Thm: Assume $H^{p,0}(X) = 0$ for $p > 0$ $\stackrel{> r}{}$. Assume moreover that: i) X satisfies Nilpotence conj. &

$n = \dim(X)$

ii) $\exists D \overset{\iota}{\subseteq} X$ closed subvariety of codim 1 &

a resolution $\tilde{D} \xrightarrow{\tilde{\iota}} D \subseteq X$ s.t. ① $H_B^{k-2}(\tilde{D}, \mathbb{Q}) \to H_B^k(X, \mathbb{Q})$ is surj $\forall k > 0$

(gen Hodge conj true for X in coniveau 1 $^{(r)}$)

(Hodge conj ⇒) ② $CH^k(\tilde{D} \times X)_{\mathbb{Q}} \xrightarrow{cl} Hdg^k(\tilde{D} \times X) = H_B^{2k}(\tilde{D} \times X, \mathbb{Q}) \cap H^{k,k}(\tilde{D} \times X)$ $\forall k \geq 0$

alg. subscheme of dim $\leq r$.

Then $CH_0(X)$ is supported on finitely many closed points.

proof: consider $[\Delta_X] \in \bigoplus_{k \geq 0} H_B^k(X, \mathbb{Q}) \otimes H^{2n-k}(X, \mathbb{Q})$ & $[\Delta_X] = [X \times \{x\}]$ mod $\bigoplus_{k > 0}$ same sum

in particular $[\Delta_X] \in Hdg^n(X \times X)$ & $[\Delta_X - X \times \{x\}] \in Hdg^n(X \times X)$

But since $k > 0 \Rightarrow$

$$\begin{array}{ccc} H_B^{k-2}(\tilde{D}, \mathbb{Q}) \otimes H_B^{2n-k}(X, \mathbb{Q}) & \twoheadrightarrow & H_B^k(X, \mathbb{Q}) \otimes H_B^{2n-k}(X, \mathbb{Q}) \\ \cap & & \cap \\ H_B^{2n-2}(\tilde{D} \times X, \mathbb{Q}) & \xrightarrow{(\tilde{\iota}, id)_*} & H_B^{2n}(X \times X, \mathbb{Q}) \end{array}$$

which respects Hodge structure.

$\Rightarrow \exists \beta \in Hdg^{n-1}(\tilde{D} \times X, \mathbb{Q})$: $[\Delta_X - X \times \{x\}] = (\tilde{\iota}, id)_* \beta$.

② $\Rightarrow \beta = cl(Z)$ $Z \in CH^{n-1}(\tilde{D} \times X, \mathbb{Q}) \Rightarrow \Gamma = \Delta_X - X \times \{x\} - (\tilde{\iota}, id)_* Z$ ($\Gamma \sim_{hom} 0$)

But Γ_* acts as id on $CH_0(X)_{hom}$. (bc $(\tilde{\iota}, id)_* Z_* = 0$ on $CH_0(X)$ bc

$(\tilde{\iota}, id)_* Z$ is supported on $D \times X$ w/ $D \subsetneq X$)

$\Rightarrow CH^n(X)_{\mathbb{Q}} \hookrightarrow H_B^{2n}(X, \mathbb{Q}) \simeq \mathbb{Q}$ .. $CH_0(X)^{\deg(0)} \twoheadrightarrow Alb_X(k)$ & Roitman's thm

${}_{tors}CH_0(X) \hookrightarrow {}_{tors}Alb_X(k)$

$\Rightarrow$ so the torsion is 0. $\Rightarrow CH_0(X) \simeq \mathbb{Z}$ ✓ .