GENERALIZED DECOMPOSITION OF THE DIAGONAL AND GENERALIZED BLOCH CONJECTURE

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Abstract. We will discuss an strength of the decomposition of the diagonal, due to Paranjape and Laterveer. This technical tool will give us a better description of the pieces appearing in the decomposition, at the expense of imposing stronger assumptions. Informally, these conditions require that the Chow groups are "small enough" to be determined by its cohomology class.

Our main application will be to discuss some vanishing of Hodge numbers implied by the small Chow groups assumption. We will relate this to the Bloch conjecture for surfaces, stating a higher dimensional generalization.

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Contents

1. Introduction

During the last couple of lectures we have discussed the following theorem[1] and its applications.

Theorem 1 (Block–Srinivas, 1983, [Voi14, 3.10])**.** *Let X be a smooth projective variety of dimension n. Assume that $\mathrm{CH}_0(X)$ is supported on some closed algebraic subset $W \subset X$. Then we can write*

$$m\Delta_X = Z_0 + Z' \tag{1}$$

on $\mathrm{CH}^n(X \times X)$, where $m > 0$ is an integer, and Z_0, Z' are codimension n cycles satisfying

$$\operatorname{Supp} Z_0 \subseteq X \times W, \qquad \operatorname{Supp} Z' \subseteq T \times X,$$

for some $T \subsetneq X$.

Informally, using (1) we can "decompose" the diagonal into two pieces, supported on a product smaller than $X \times X$. This was used in the last couple of talks by looking at the induced correspondences: the equality (1) induces

$$m\,\mathrm{id} = [Z_0]^* + [Z']^* \colon H^n(X, \mathbb{Z}) \to H^n(X, \mathbb{Z}).$$

[1]The theorem itself had slightly different notation. This is on purpose, as we will see in a second.

Now, the correspondences $[Z_0]^*$ and $[Z']^*$ factor through pieces of smaller dimension, which allowed us to get results about the *coniveau* of $H^n(X,\mathbb{Z})$. However, applications of this result are a bit tricky, as we have no control on T.

Our goal today is to discuss a generalization of these ideas, due to Paranjape (1994) and Laterveer (1996). At the expense of replacing (1) with more pieces (and by imposing stronger assumptions), this will allow us to have a better control of the support of T. In particular, we will be able to give better results about the coniveau of $H^*(X,\mathbb{Z})$.

2. Generalized decomposition of the diagonal

Let us start with the main technical tool we will use in this talk.

Theorem 2 (Paranjape, 1994; Laterveer, 1996, cf. [Voi14, 3.18])**.** *Let X be a smooth projective variety of dimension n, and let $c \geq 1$ be an integer. Assume that for $k < c$, the cycle class maps*

$$\mathrm{cl}\colon \mathrm{CH}_k(X)\otimes\mathbb{Q} \to H^{2n-2k}(X,\mathbb{Q}) \tag{2}$$

are injective. Then there exists a decomposition

$$m\Delta_X = Z_0 + \cdots + Z_{c-1} + Z' \tag{3}$$

in $\mathrm{CH}^n(X\times X)$, where $m>0$ is an integer, $\operatorname{Supp} Z_i \subseteq W_i'\times W_i$ with $\dim W_i' = n-i$ and $\dim W_i = i$, and $\operatorname{Supp} Z' \subseteq T\times X$, for some $T\subseteq X$ of codimension $\geq c$.

Before we discuss the proof of the theorem, let us try to understand the injectivity assumption. For $k=0$, the injectivity of the map

$$\mathrm{cl}\colon \mathrm{CH}_0(X)\times\mathbb{Q}\to H^{2n}(X,\mathbb{Q})$$

is already a strong assumption: it means that all points on X are rationally equivalent. It is easy to produce examples where this cannot hold, such an elliptic curve. On the opposite, we may expect that this map is injective when X has many rational curves, such as rational or Fano varieties.

In some sense, the condition (2) implies that $\mathrm{CH}_k(X)$ is "small" for $k<c$, as it fits inside $H^{2n-2k}(X,\mathbb{Q})$. Recall here that CH_k might be infinite dimensional in general, while $H^{2n-2k}(X,\mathbb{Q})$ is finite dimensional.

As a nice observation, condition (4) can be rephrased in terms of the usual class map

$$\mathrm{cl}\colon \mathrm{CH}_k(X)\to H^{2n-2k}(X,\mathbb{Q}).$$

The condition implies that all the elements of $\mathrm{CH}_k(X)_{\mathrm{hom}}$, the kernel of cl, are torsion.

The proof of this result will require the following consequence of the Bloch–Srinivas theorem.

Lemma 3. *Let $\Gamma\in\mathrm{CH}^k(X\times Y)$ and $X'\subset X$ be given. Assume that for every $x\in X$, the cycle $\Gamma^*(x)\in\mathrm{CH}^k(Y)$ restricts to zero in $\mathrm{CH}^k(X-X')$. Then we have a decomposition*

$$m\Gamma = \Gamma' + \Gamma'',$$

where Γ' is supported in $X'\times Y$, and Γ'' is supported in $X\times Y'$ for some proper closed subset $Y'\subsetneq Y$.

Proof. (of Theorem 2) We will proceed by induction on c. Let us start by looking at the case $c = 1$. Here, the assumption (2) implies that $\mathrm{CH}_0(\{pt\}) \to \mathrm{CH}_0(X)$ is surjective. Thus, the required result follows directly by the usual decomposition of the diagonal.

Now, let $c \geq 2$ be given. By induction, there is a decomposition

$$m\Delta_X = Z_0 + \cdots + Z_{c-2} + Z', \tag{4}$$

where $m > 0$ is an integer, Z_i is supported inside $W_i' \times W_i$ with $\dim W_i' = n - i$ and $\dim W_i = i$, and Z' is supported in $T \times X$, where T is a closed algebraic subset of codimension $\geq c - 1$. Let us assume that T is pure of dimension $c - 1$, for the sake of simplicity.[2]

Let $\tilde{T}$ be a resolution of singularities of T, and let $\tau \colon \tilde{T} \to X$ be the induced map. Lift the cycle Z' to some $\tilde{Z}' \subset \tilde{T} \times X$, satisfying $(\tau, \mathrm{id})_* \tilde{Z}' = Z'$. This way, the cycle $\tilde{Z}'$ has codimension $(n - c + 1)$ inside $\tilde{T} \times X$, and it induces a morphism $\tilde{Z}'_* \colon \mathrm{CH}_0(\tilde{T}) \to \mathrm{CH}_{c-1}(X)$.

Now, recall that the class map is compatible with correspondences. Thus, $\tilde{Z}'_*$ maps $\mathrm{CH}_0(\tilde{T})_{\mathrm{hom}}$ (the kernel of the class map) to $\mathrm{CH}_{c-1}(X)_{\mathrm{hom}}$. By assumption, this kernel is torsion. This way, for every integer a look at

$$\{t \in \tilde{T} : a\tilde{Z}'_*(t - t_0) = 0\}$$

for some fixed point t_0. These are closed subsets, and their union is all of $\tilde{T}$. By Baire's category theorem, there is some $a > 0$ such that $a\tilde{Z}'_* = 0$ on $\mathrm{CH}_0(\tilde{T})_{\mathrm{hom}}$.

Now, for each component $\tilde{T}_i$ of $\tilde{T}$, pick a point $t_i \in \tilde{T}_i$. Write $W_i = \tilde{Z}'_*(t_i) \in \mathrm{CH}_{c-1}(X)$, so that the cycle

$$Z'' = a\left(\tilde{Z}' - \sum_i \tilde{T}_i \times W_i\right) \subset \tilde{T} \times X$$

satisfies the following property. The induced morphism $Z''_* \colon \mathrm{CH}_0(\tilde{T}) \to \mathrm{CH}_{c-1}(X)$ is the zero morphism. In fact, note that $Z''_*(t - t_i) = 0$ if t lies in $\tilde{T}_i$, which proves this statement.

Let us re-write what we have proven here. The cycle $a\tilde{Z}'$ satisfies that for every $t \in \tilde{T}$, the induced element $a\tilde{Z}'_*(t)$ restricts to zero in $\mathrm{CH}_{c-}(X - W)$ for $W = \bigcup_i W_i$. This way, by Lemma 3 we get a subset $\tilde{T}' \subsetneq \tilde{T}$ of codimension[3] ≥ 1, an integer $b > 0$, and a decomposition

$$ab\tilde{Z}' = \tilde{Z}_{c-1} + \tilde{Z}'',$$

so that $\tilde{Z}_{c-1}$ is supported on $\tilde{T} \times W_i$, and $\tilde{Z}''$ in $\tilde{T}' \times X$. Pushforward these cycles to $T \times X$ and use the inductive assumption to conclude. □

As an important remark, note that nothing is said about the irreducibility of W_i' or W_i. However, from the proof is clear that each Z_i is a sum of $W_{ij}' \times W_{ik}$, for some irreducible components of W_i' and W_i.

Before we move on, let us show an application of this theorem. Informally, it says that injectivity implies surjectivity for the class map.

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[2]For the general case we split Z' into two pieces: one supported on a piece pure of codimension $c - 1$, and one on codimension $\geq c - 2$. The second one will become part of the new Z' later.

[3]A priori, it seems that $\tilde{T}'$ is only of codimension ≥ 1 on one component. However, as $\tilde{T}'$ is a disjoint union of its irreducible components, it is clear that we can take $\tilde{T}'$ to be of codimension ≥ 1 to conclude.

Theorem 4 (Lewis, 1995; cf. [Voi14, 3.9])**.** *Let X be a smooth, projective, complex variety. Assume that the cycle class* $\mathrm{cl}\colon \mathrm{CH}^i(X)_{\mathbb{Q}} \to H^{2i}(X,\mathbb{Q})$ *is injective for all $i \geq 0$. Then $H^{2i+1}(X,\mathbb{Q}) = 0$ for all $i \geq 0$, and the cycle class map is an isomorphism for all $i \geq 0$.*

Proof. This is a direct application of Theorem 2. Note that we have a *complete* decomposition of the diagonal

$$\Delta_X = \sum_i a_i V_i \times W_i$$

in $\mathrm{CH}^n(X \times X)_{\mathbb{Q}}$. Here V_i and W_i are some irreducible subsets of dimensions d_i and $n - d_i$, respectively.

This way, we get an induced cohomological decomposition. If $\alpha \in H^*(X,\mathbb{Q})$, we get

$$\alpha = [\Delta_X]_* \alpha = \sum_i a_i \langle \alpha, [V_i] \rangle [W_i].$$

In particular, we get that all odd degree cohomology vanishes, as the right hand side is supported on even cohomology. Moreover, this proves that even cohomology is generated by algebraic cycles, as claimed. □

3. Generalized Bloch conjecture

Let us start recalling the classical *Bloch conjecture.* Let S be a smooth, projective surface. Recall from Mumford's theorem the following consequence.

Proposition 5. *Assume that* $\mathrm{CH}_0(S) \cong \mathbb{Z}$*. Then* $p_g(S) = H^0(S, \Omega_S^2)$ *is trivial.*

Proof. This is a direct consequence of Mumford's theorem, as then the pushforward map $j_*\colon \mathrm{CH}_0(pt) \to \mathrm{CH}_0(S)$ is surjective. □

Conjecture 6 (Bloch, 1980)**.** *Assume that* $p_g(S) = 0$*. Then* $\mathrm{CH}_0(S) \cong \mathbb{Z}$*.*

Note that under these assumptions the class map $\mathrm{CH}_0(S)_{\mathbb{Q}} \to H^4(X,\mathbb{Q})$ is injective. Informally, Bloch's conjecture predicts that the 0-cycle class map is injective, provided that we have $h^{2,0} = 0$.

Now, note that $H^{0,2} \oplus H^{0,2} \subset H^2(S,\mathbb{C})$ corresponds exactly to the *transcendental* part of the cohomology, i.e. the orthogonal to the span of algebraic curves in $H^2(S,\mathbb{C})$. (This is a consequence of the fact that $H^{1,1}$ is spanned by algebraic curves, by the Lefschetz theorem on $(1,1)$ cycles.) It is natural now to use these ideas to higher dimensional cycles.

Conjecture 7 (Generalized Bloch conjecture, cf. [Voi14, 3.21])**.** *Let X be a smooth projective variety of dimension m. Assume that $H^{p,q}(X) = 0$ for $p \neq q$ and $p < c$. Then for any integer $i < c$, the cycle class map*

$$\mathrm{cl}\colon \mathrm{CH}_i(X)_{\mathbb{Q}} \to H^{2m-2i}(X,\mathbb{Q})$$

is injective.

An alternative formulation of this conjecture can be perform assuming the generalized Hodge conjecture. The advantage is that there is no mention of Hodge structures.

Conjecture 8 ([Voi14, 3.23])**.** *Assume that the transcendental cohomology of X is supported on a closed algebraic subset of codimension $\geq c$. Then, for any $i < c$, the cycle class map*

$$\mathrm{cl}\colon \mathrm{CH}_i(X)_{\mathbb{Q}} \to H^{2m-2i}(X, \mathbb{Q})$$

is injective.

Similar to Bloch conjecture, the biggest piece of evidence comes from the fact that its converse is true.

Theorem 9 (Laterveer 1996, Lewis 1995, Paranjape 1994, Schoen 1993; cf. [Voi14, 3.20])**.** *Let X be a smooth projective variety of dimension m. Assume that the cycle class map*

$$\mathrm{cl}\colon \mathrm{CH}_i(X)_{\mathbb{Q}} \to H^{2m-2i}(X, \mathbb{Q})$$

is injective for $i < c$. Then we have $H^{p,q} = 0$ for $p \neq q$ and $p < c$.

Once again, the key idea will turn out to be interpreting the (generalized) decomposition of the diagonal as a decomposition of the identity operator in cohomology.

Proof. (of Theorem 9) WIP □

References

[Voi14] Claire Voisin, *Chow rings, decomposition of the diagonal, and the topology of families*, Annals of Mathematics Studies, vol. 187, Princeton University Press, Princeton, NJ, 2014. MR3186044

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