1 02/23 Matthew (Mixed Hodge structures and coniveau)

As always, all varieties are over $\mathbb{C}$. Recall from last time that for X a smooth projective variety, there is a natural Hodge decomposition

$$H^m(X) \otimes \mathbb{C} = \bigoplus_{p+q=m} H^{p,q}$$

such that $\overline{H^{p,q}} = H^{q,p}$, which we call a pure Hodge structure of weight n. Here, $H^m(X) \coloneqq H^m(X, \mathbb{Q})$.

Concisely, we say the functor $X \mapsto H^m(X)$ is a contravariant functor from smooth projective varieties to the category of pure Hodge structures of weight m.

Today, I want to talk about what happens when the source category consists of quasi-projective varieties. To do so, I'll start with the abstract notion of a mixed Hodge structure.

Definition 1. A rational **mixed Hodge structure** (MHS) is given by a $\mathbb{Q}$-vector space H along with two filtrations: 1. an increasing **weight filtration** $W_i H$ and 2. a decreasing **Hodge filtration** $F^k H_{\mathbb{C}}$. These filtrations are compatible in the sense that the induced Hodge filtration on each $\operatorname{Gr}_i^W H$ is a Hodge structure of weight i.

We also assume $F^{\text{small}} H_{\mathbb{C}} = H_{\mathbb{C}}$, $F^{\text{large}} H_{\mathbb{C}} = 0$, $W_{\text{small}} H = 0$, and $W_{\text{big}} H = H$.

Remark 2. Voisin wants the Hodge structure to have weight $n + i$ instead of i, but this doesn't seem standard.

Explicitly, this means

$$\operatorname{Gr}_i^W H_{\mathbb{C}} = \underbrace{F^k \operatorname{Gr}_i^W H_{\mathbb{C}}}_{\operatorname{im}\left(F^k H_{\mathbb{C}} \cap W_i H_{\mathbb{C}} \to \operatorname{Gr}_i^W H_{\mathbb{C}}\right)} \oplus \overline{F^{i-k+1} \operatorname{Gr}_i^W H_{\mathbb{C}}}.$$

Deligne proved the following:

Theorem 3. *Let X be a quasi-projective variety. Then, $H^m(X)$ has a natural MHS. Moreover, the weight filtration has the following properties:*

(i) The weight filtration looks like $0 = W_{-1} \subset W_0 \subset \cdots \subset W_{2m} = H^m(X)$.

(ii) If X is smooth and projective, then $0 = W_{m-1} \subset W_m = H^m(X)$, i.e. $H^m(X, \mathbb{Q})$ is of pure weight m.

(iii) If X is projective (but not smooth), then $0 = W_{-1} \subset W_0 \subset \cdots \subset W_m = H^m(X)$.

(iv) If X is smooth (but not projective), then $0 = W_{m-1} \subset W_m \subset \cdots \subset W_{2m} = H^m(X)$. Also, for any smooth compactification $j\colon X \subset \overline{X}$, we have $W_m = j^* H^m\left(\overline{X}\right)$.

(v) For any morphism f of varieties $X \to Y$, we have f^* sends W_i to W_i and the pure Hodge structures of weight i on the graded pieces.

Let's do an extremely basic example.

Example 4. Let X_1 and X_2 be smooth projective varieties that intersect transversely. We'll compute the weight filtration of $H^m(X)$, where $X = X_1 \cup X_2$.

We have the Mayer-Vietoris sequence

$$\overset{\beta_{m-1}}{\to} H^{m-1}(X_1 \cap X_2) \overset{\delta}{\to} H^m(X) \overset{\alpha}{\to} H^m(X_1) \oplus H^m(X_2) \overset{\beta_m}{\to} H^m(X_1 \cap X_2) \to$$

Since $X_1 \cap X_2, X_1$, and X_2 are smooth projective, their cohomologies are pure Hodge structures. In particular, the first term has pure weight $m-1$, and the last two have pure weight m.

Assuming that all of the maps are maps of MHS, it follows the weight filtration on $H^m(X)$ is given by

$$W_{m-2} = 0,$$

$$W_{m-1} = \operatorname{im}\delta,$$

and

$$W_m = H^m(X).$$

This makes sense because the quotients $W_m/W_{m-1} = \ker\beta_m$ and $W_{m-1}/W_{m-2} = \operatorname{im}\delta \cong \operatorname{coker}\beta_{m-1}$ are Hodge structures of weight m and $m-1$, respectively.

Definition 5. A **morphism** of MHS is one that respects filtrations.

Deligne showed the following:

Theorem 6. *A morphism* $\alpha: (H, W, F) \to (H', W', F')$ *of MHS is strict with respect to both filtrations, i.e.* $\operatorname{im}\alpha \cap W'_i H'_{\mathbb{C}} = \alpha(W_i H_{\mathbb{C}})$ *and* $\operatorname{im}\alpha \cap F'^i H'_{\mathbb{C}} = \alpha(F^i H_{\mathbb{C}})$.

Proof. For any MHS (H, W, F), there is a decomposition

$$H_{\mathbb{C}} = \bigoplus_{p,q} H^{p,q}$$

with $H^{p,q} \subset F^p H_{\mathbb{C}} \cap W_{p+q} H_{\mathbb{C}}$, such that $H^{p,q}$ can be identified with $H^{p,q}\left(\operatorname{Gr}^W_{p+q} H_{\mathbb{C}}\right)$ under the projection $W_{p+q} H_{\mathbb{C}} \to \operatorname{Gr}^W_{p+q} H_{\mathbb{C}}$. Moreover, this decomposition is respected by morphisms of MHS.

Now, to show the claim (at least for weight filtration), let $\ell' \in \alpha(H_{\mathbb{C}}) \cap W'_i H'$, let us write $\ell' = \alpha(\ell)$ and decompose ℓ as $\sum \ell^{p,q}$ as above. Then, $\alpha(\ell) = \sum \alpha(\ell^{p,q})$ with $\alpha(\ell^{p,q}) \in H'^{p,q}$.

Moreover, $\ell' \in W'_i H'_{\mathbb{C}}$, so $\alpha(\ell^{p,q}) = 0$ for $p+q > i$, i.e.

$$\ell' = \alpha\left(\sum_{p+q \le i} \ell^{p,q}\right) \in \alpha(W_i H_{\mathbb{C}}).$$

☺

Here is a remarkable application of MHS, known as the global invariant cycle theorem:

Theorem 7. *Let $\overline{X}$ be a smooth projective variety, $\overline{Y}$ a smooth projective curve, and $\overline{f}\colon \overline{X} \to \overline{Y}$ a morphism. Let $S \subset \overline{Y}$ be the set of critical values, $Y = \overline{Y} \backslash S$, and $X = \overline{f}^{-1}(Y)$, so that $f\colon X \to Y$ is smooth and projective.*

Let $y \in Y$ and X_y be the fiber. Then, the image of $H^m\left(\overline{X}\right) \to H^m\left(X_y\right)$ coincides with $H^m\left(X_y\right)^{\pi_1(Y,y)}$.

Proof. Deligne proved $Rf_*\mathbb{Q}_X = \bigoplus R^m f_*\mathbb{Q}_X[-m]$ in $D_c^b(Y)$ (non-canonically), which means we have a surjection of MHS

$$H^m(X) \to H^0\left(Y, R^m f_*\mathbb{Q}_X\right).$$

Moreover, the latter is the $\pi_1(Y,y)$-invariants of the stalk of the lisse sheaf $R^m f_*\mathbb{Q}_X$ at y, which is $H^m\left(X_y\right)^{\pi_1(Y,y)}$.

Moreover, $H^m\left(\overline{X}\right)$ coincides with $\mathrm{Gr}_m^W\left(H^m(X)\right)$ (by construction), so by strictness since $H^m\left(X_y\right)$ is a pure Hodge structure of weight m, it follows that $H^m\left(\overline{X}\right) \to H^m\left(X_y\right)$ coincides with $H^m\left(X_y\right)^{\pi_1(Y,y)}$. ☺

Next, we'll talk about (Hodge) coniveau, which will allow us to state a generalized version of the Hodge conjecture.

Definition 8. A weight k (pure) Hodge structure $(L, L^{p,q})$ has **coniveau** $c \leq k/2$ if the Hodge decomposition of $L_{\mathbb{C}}$ has the form

$$L_{\mathbb{C}} = L^{k-c,c} \oplus L^{k-c-1,c+1} \oplus \cdots \oplus L^{c,k-c}$$

with $L^{k-c,c} \neq 0$.

Theorem 9. *If X is a smooth complex projective variety and $i\colon Y \hookrightarrow X$ a closed subvariety of codimension c. Let $j\colon X\backslash Y \subset X$ be the open inclusion and j^* be the pullback $H^k(X) \to H^k(X\backslash Y)$.*

Then $\ker j^*$ *is a sub-Hodge structure of coniveau $\geq c$ of $H^k(X)$.*

Proof. The key point is to use the strictness of the weight filtration for morphisms between MHS.

Let $n = \dim X$. Choose $\widetilde{Y} \to Y$ a desingularization with pure complex dimension $n - c$.

We have a long exact sequence

$$\to H_{2n-k}^{\mathrm{BM}}(Y)(-k) \to H^k(X) \to H^k(U) \to,$$

so $\ker j^* = \mathrm{im}\left(H_{2n-k}^{\mathrm{BM}}(Y)(-n) \to H^k(X)\right)$. Here, we have $H_m^{\mathrm{BM}}(X) \coloneqq R^{-m}\Gamma(X, D_X)$ and for a smooth variety X, the Poincare homomorphism (induced by $\mathbb{Z}_X[2d](d) \to D_X$) $H^m(X) \to H_{2d-m}^{\mathrm{BM}}(X)(-d)$ is an isomorphism.

The map $H_{2n-k}^{\mathrm{BM}}(Y)(-n) \to H^k(X)$ is a morphism of MHS, and the RHS is moreover a pure Hodge structure of weight k. By strictness, the image must be the same as the image of $W_k H_{2n-k}^{\mathrm{BM}}(Y)(-n) \to H^k(X)$.

Apparently, by construction, $W_k H^{\mathrm{BM}}_{2n-k}(Y)(-n)$ coincides with the image of $H^{\mathrm{BM}}_{2n-k}(\widetilde{Y})(-n) \to H^{\mathrm{BM}}_{2n-k}(Y)(-n)$. Since $\widetilde{Y}$ is smooth, we have $H^{k-2c}\left(\widetilde{Y}\right) = H^{\mathrm{BM}}_{2n-k}\left(\widetilde{Y}\right)(-n+c)$.

In other words, $\ker j^*$ is the same as the image of $H^{k-2c}\left(\widetilde{Y}\right)(-c) \to H^k(X)$. The result follows. ☺

Grothendieck conjectured the following, known as the **generalized Hodge conjecture**:

Conjecture 10. *Let $L \subset H^k(X)$ be a rational sub-Hodge structure of coniveau at least c. Then there is a closed algebraic subset $Z \subset X$ of codimension c such that L vanishes under the restriction $H^k(X) \to H^k(X \backslash Z)$.*

In the setting where $k = 2c$, the usual Hodge conjecture implies the generalized Hodge conjecture. Indeed, we have $L_{\mathbb{C}} = L^{c,c}$ with L conisisting of Hodge classes. By the Hodge conjecture, we can find codimension c cycles $Z_1, \ldots, Z_N$ of X such that L is generated by $[Z_1], \ldots, [Z_N]$. Then, L vanishes on the complement of the supports of $Z_1, \ldots, Z_N$.

For $k = 2c+1$, where $L_{\mathbb{C}} = L^{c+1,c} \oplus L^{c,c+1}$, a much more difficult argument showing Hodge implies generalized Hodge exists somewhere—read Voisin!