PURE HODGE STRUCTURES AND STANDARD CONJECTURES

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Abstract. In algebraic geometry, Hodge structures are one of the key tools to understanding cohomology of algebraic varieties. Their definition is motivated by the decomposition of the singular cohomology of a compact Kähler manifold, first studied by William Hodge in the 1930's. Further developments have extended similar results to non-compact manifolds, singular varieties, and so on.

In this talk we will focus on pure Hodge structures, discussing their basic properties and key examples. This will allow us to state the so-called "standard conjectures", a set of problems regarding the interplay between cohomology classes and algebraic cycles.

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1. Pure Hodge structures

Our starting point is the following definition.

Definition 1 ([Voi14, 2.20], cf. [Voi02, 7.4])**.** Let k be an integer. A *weight k rational (pure) Hodge structure* $(V, V^{p,q})$ consists on a $\mathbb{Q}$-vector space V, and a $\mathbb{C}$-vector space decomposition

$$V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q},$$

satisfying $V^{p,q} = \overline{V^{q,p}}$.

We point out that this definition also makes sense replacing $\mathbb{Q}$ with $\mathbb{Z}$ or $\mathbb{R}$, in which case we talk about *integral* (resp. *real*) Hodge structures. Unless explicitly mentioned, all Hodge structures will be over $\mathbb{Q}$.

Example 2. Let us present two simple but interesting examples.

(1) Let m be an integer. Take $V = \mathbb{Q}$, with the decomposition $V^{-m,-m} = \mathbb{C}$ (and zero otherwise). This is a rational Hodge structure of weight $-2m$.

(2) Let V be a vector space, and let $I\colon V \to V$ be a linear transformation satisfying $I^2 = -1$. (The key example to keep in mind is the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

as we will see later.) We extend scalars to get $I_{\mathbb{C}}\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$, which is now a diagonalizable operator. Let $V^{1,0}$ (resp. $V^{0,1}$) be the eigenspace of eigenvalue i (resp. $-i$). Then $\overline{V^{1,0}} = V^{0,1}$, and so this defines a weight 1 rational Hodge structure on V.

(3) The previous example can be generalized a bit: given $0 \leq k \leq \dim V$, the vector space $\wedge^k V$ carries a weight k rational Hodge structure. In fact, we have

$$\wedge^k V \otimes \mathbb{C} \cong \bigoplus_{p+q=k} \wedge^p V^{1,0} \otimes \wedge^q V^{0,1}.$$

We use this to set $(\wedge^k V)^{p,q} := \wedge^p V^{1,0} \otimes \wedge^q V^{0,1}$.

Example 3. Some linear algebra operations can be extended naturally to Hodge structures. For instance, if V and W are two Hodge structures of the same weight, $V \oplus W$ carries naturally a Hodge structure. Similarly, we can take the tensor product of two Hodge structures, whose weight will be the sum of the previous two.

Remark 4 ([Voi14, p. 24]). Given a Hodge structure $(V, V^{p,q})$ of weight k, we define a filtration of $V_{\mathbb{C}}$ via

$$F^p V_{\mathbb{C}} := \bigoplus_{r \geq p} V^{r,k-r}.$$

This is a decreasing filtration, satisfying

$$V^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}.$$

We could have defined the Hodge decomposition in terms of a filtration instead. This might look weird on a first glance; however, in some cases it is more natural.

1.1. **Motivation: Hodge decomposition.** Our definition is motivated by the *Hodge decomposition.* Informally, given a compact Kähler manifold X, this results endows $H^k(X, \mathbb{Q})$ with a weight k Hodge structure. We follow the discussion on [Voi02, §6].

Let X be a compact Kähler manifold, and fix a metric on X. We have an isomorphism $H^k(X, \mathbb{C}) \cong \mathcal{H}^k(X)$, the set of complex valued harmonic forms for the Laplacian associated to this metric. This way, by separating the forms on types, we get an induced decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=\mathbb{C}} H^{p,q},$$

where $H^{p,q}$ are the set of harmonic forms of type (p, q). It turns out that this decomposition does not depend on the metric (cf. [Voi02, 6.11]), and that the components $H^{p,q}$ satisfy the relation

$$\overline{H^{p,q}} = H^{q,p},$$

where the conjugate is with respect to the $\mathbb{C}$-vector space structure on $H^k(X, \mathbb{C}) = H^k(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$.

Putting all together, we get the following result.

Proposition 5 (cf. [Voi14, pp. 24–25; Voi02, pp. 142–143]). *Let X be a compact Kähler manifold, and let $0 \leq k \leq 2\dim X$. Then $H^k(X,\mathbb{Q})$ carries a weight k rational Hodge structure.*

Interestingly, the filtration associated to this Hodge structure can be understood directly from the sheaves of holomorphic forms. To do this, recall that by the Poincaré lemma we have that the complex

$$\underline{\mathbb{C}} \to \mathscr{O}_X \xrightarrow{\partial} \Omega^1_X \xrightarrow{\partial} \Omega^2_X \xrightarrow{\partial} \cdots$$

is exact. (Here either X is a complex manifold, or the analytic space associated to a smooth variety.) This way, the cohomology $H^k(X,\mathbb{C})$ is identified with the hypercohomology of the complex

$$\Omega^\bullet_X = [\mathscr{O}_X \xrightarrow{\partial} \Omega^1_X \xrightarrow{\partial} \Omega^2_X \xrightarrow{\partial} \cdots]$$

Now, let $F^p\Omega^\bullet_X = \Omega^{\bullet \geq p}_X$ be the *bête* (or *naive*) filtration. This is a decreasing filtration, endowed with maps to $\Omega^\bullet_X$. Taking cohomology we get the following

Proposition 6 (Frölicher spectral sequence, cf. [Voi02, §8.3.3]). *There is a spectral sequence*

$$E_1^{p,q} = H^q(X,\Omega^p_X) \Rightarrow H^{p+q}(X,\Omega^\bullet_X).$$

If X is compact Kähler (e.g. if X comes from a smooth, projective variety), then this sequence degenerates at E_1. The induced filtration of $H^{p+q}(X,\Omega^\bullet_X)$ is exactly the Hodge filtration.

This approach has some extra advantages. For instance, it allows us to construct "Hodge filtrations in families", cf. [Voi02, §10.2.1.]

2. Polarizations

Definition 7 ([Voi14, 2.21]). Let $(V, V^{p,q})$ be a weight k rational Hodge structure. A *polarization* on V is a non-degenerate pairing $(\cdot,\cdot)$ on V, symmetric (resp. skew symmetric) for k even (resp k odd), satisfying the following two conditions.

The Hermitian pairing $H(a,b) = i^k(a,\bar{b})$ on $V_\mathbb{C}$ satisfies the following.

1. (First Hodge–Riemann bilinear relations) The Hodge decomposition of V is orthogonal with respect to H.
2. (Second Hodge–Riemann bilinear relations) The restriction $H|_{V^{p,q}}$ is definite of sign $(-1)^p$.

One of the nice properties of having a polarization is that we get decomposition of Hodge structures into ones without sub-Hodge structures.

Theorem 8 ([Voi14, 2.22]). *Let $(V, V^{p,q})$ be a rational polarized Hodge structure, and let $U \subset V$ be a sub-Hodge structure. Then V decomposes as a direct sum $V = U \oplus W$.*

Proof. Note that the restriction of $(\cdot,\cdot)_U$ is a non-degenerate pairing. In fact, each $U^{p,q} \subset V^{p,q}$ is a non-degenerate subspace (as H is definite on $V^{p,q}$), and the $U^{p,q}$ are orthogonal to each other.

We now let $W = U^\perp$. We note that

$$W \supseteq \bigoplus_{p+q=k} (U^{p,q})^\perp \cap V^{p,q},$$

and by dimension counting these are actually equal. It follows that W is a sub-Hodge structure, and that $V = U \oplus W$. □

Corollary 9 ([Voi02, 2.24]). *Let $\phi\colon V \to W$ be a morphism of rational Hodge structures. Then $\ker\phi$ and $\operatorname{Im}\phi$ are sub-Hodge structures of V and W, respectively.*

Moreover, assume that V is polarized. Then V contains a sub-Hodge structure V' so that ϕ induces an isomorphism $V' \to \operatorname{Im}\phi$.

Proof. The first part is clear. For the second, we just take $V' = (\ker\phi)^{\perp}$. □

The key example for polarized Hodge structures comes from smooth, projective varieties. Let X be a smooth, projective variety of dimension n, and pick an ample line bundle $\mathscr{L}$. If $\omega = c_1(\mathscr{L}) \in H^2(X, \mathbb{Q})$ is the first Chern class, then $\omega \cup -$ induces[1] an operator

$$L\colon H^k(X, \mathbb{Q}) \to H^{k+2}(X, \mathbb{Q}),$$

called the *Lefschetz operator*. For $k \leq n$, it induces isomorphisms

$$L^{n-k}\colon H^k(X, \mathbb{Q}) \xrightarrow{\sim} H^{2n-k}(X, \mathbb{Q}),$$

which is known as the *hard Lefschetz property.*

Using the Lefschetz operator, we get a decomposition

$$H^k(X, \mathbb{Q}) = \bigoplus_{r : k-2r \geq 0} L^k H^{k-2r}(X, \mathbb{Q})_{\text{prim}}, \tag{1}$$

where $H^i(X, \mathbb{Q})_{\text{prim}}$ is the kernel of L^{n-k+1}.

At last, we have an induced pairing

$$(\alpha, \beta)_k = \int_X \omega^{n-k} \cup \alpha \cup \beta.$$

on $H^k(X, \mathbb{Q})$.

Lemma 10 ([Voi02, 6.31]). *For $k \leq n$, the decomposition* (1) *is orthogonal with respect to the Hermitian pairing H_k induced by $(-, -)_k$.*

Up to some signs, the previous pairing will give us a polarization on each primitive piece.

Theorem 11 ([Voi02, 6.31, 6.32]). *On each piece $L^r H^{k-2r}(X, \mathbb{C})_{\text{prim}}$, the pairing H_r induces $(-1)^r H_{k-2r}$. Also, the subspaces $H^{p,q}(X)$ are orthogonal. Moreover, the form $(-1)^{k(k-1)/2} i^{p-q-k} H_k$ is positive definite on $H^{p,q}_{\text{prim}} := H^k(X, \mathbb{C})_{\text{prim}} \cap H^{p,q}(\mathbb{C})$.*

3. Hodge classes

We start with a completely abstract definition.

Definition 12 ([Voi14, 2.23]). Let $(V, V^{p,q})$ be a weight $2k$ rational Hodge structure. A *Hodge class* is a class in $V \cap V^{k,k}$.

[1] Note that [Voi02] is using the wedge product, while [Voi14] uses the cup product. These two operations can be identified, cf. [Voi02, 5.29]

The main example of Hodge classes comes, once again, for the scenario on complex varieties. Let X be a smooth, projective variety of dimension n. If $Z \subset X$ is a reduced, irreducible subscheme of codimension k, we defined a *cycle class* $[Z] \in H^{2k}(X, \mathbb{Q})$. Moreover, we showed that there is an induced map

$$\mathrm{cl}\colon \mathrm{CH}^k(X) \to H^{2k}(X, \mathbb{Q}).$$

These maps are compatible with the intersection product on CH, and the cup product on cohomology.

Theorem 13 ([Voi02, 11.20]). *The image of* cl *is contained in the Hodge classes. This is, for each $Z \subset X$ pure of codimension k, the associated cycle $[Z] \in H^{2k}(X, \mathbb{Q})$ lies in $\mathrm{Hdg}^{2k}(X) := H^{2k}(X, \mathbb{Q}) \cap H^{k,k}$.*

The most naive question we could ask ourselves if this map is surjective. This is a naive expectation, as $\mathrm{CH}^k(X)$ only admits integral sums[2]. The next naive question was raised (not in this exact formulation) by Sir William Hodge on the 1950 ICM.

Conjecture 14 (Hodge, 1950). *Let X be a smooth, projective variety. Any Hodge class $\alpha \in \mathrm{Hdg}^{2k}(X)$ is a linear combination with rational coefficients of Betti cycle classes of algebraic subvarieties of X, so*

$$\alpha = \sum_{i=1}^{N} a_i [Z_i], \qquad a_i \in \mathbb{Q}.$$

Not many cases are known for the Hodge conjecture. One of the few known cases is known as the *Lefschetz theorem on (1,1) classes*.

Theorem 15 (Lefschetz, 1924, cf. [Voi02, 11.30]). *The Hodge conjecture holds for $k = 1$.*

Corollary 16 ([Voi14, p. 27]). *The Hodge conjecture holds for $k = \dim X - 1$.*

Proof. (Sketch) Recall that the class map is compatible with intersections and products. Therefore, if $H \in \mathrm{CH}^1(X)$ is the divisor class of an ample divisor, then the Lefschetz operator associated to $\omega = c_1(H)$ gives the required isomorphism. □

4. Standard conjectures

Our last objective is to discuss various conjectures related to the Hodge conjecture. These carry the name *standard conjectures*. We will formulate them following the discussion of [Voi14, §2.2.3].

It is worth mentioning that these conjectures can be stated in more general scenarios. The key realization is that the only facts that we will use about the (singular) cohomology $H(-, \mathbb{C})$ are Künneth decomposition and Poincaré duality. Therefore, it will be possible to state the "standard conjectures" for ℓ-adic cohomology or crystalline cohomology, to name a few. More generally, we can state these conjectures for any *Weil cohomology theory*, for which we refer to [Kle94].

[2] And is well-know that even if we restrict ourselves to $H^{2k}(X, \mathbb{Z})$, we will not get surjectivity in general. See examples by Atiyah–Hirzebruch, and by Kollár.

To start, we let X and Y be two (smooth, projective) varieties, where $\dim X = n$. The Künneth formula and Serre duality yield isomorphisms

$$\begin{aligned} H^m(X \times Y, \mathbb{Q}) &= \bigoplus_{p+q=m} H^p(X, \mathbb{Q}) \otimes H^q(X, \mathbb{Q}) \\ &= \bigoplus_{p+q=m} \operatorname{Hom}(H^{2n-p}(X, \mathbb{Q}), H^q(X, \mathbb{Q})). \end{aligned} \tag{2}$$

We point out that the right hand side are isomorphisms of *vector spaces*. This way, we could ask ourselves what happens if we require that the induced morphism is a map of Hodge structures.

Lemma 17 ([Voi02, 11.41])**.** *Assume that $m = 2r$ is even, and let $n = \dim X$. A class*

$$\alpha \in H^p(X, \mathbb{Q}) \otimes H^q(Y, \mathbb{Q}) \subset H^m(X \times Y, \mathbb{Q})$$

is a Hodge class if and only if the associated morphism

$$\tilde{\alpha} \colon H^{2n-p}(X, \mathbb{Q}) \to H^q(Y, \mathbb{Q})$$

is a morphism of Hodge structures (of bidegree $(r - n, r - n)$).

Proof. (Sketch) Note that if $\alpha = \beta^{k,\ell} \otimes \gamma^{k',\ell'}$ is an element of $H^p(X, \mathbb{Q}) \otimes H^q(Y, \mathbb{Q})$, then the induced operator $\tilde{\alpha}$ with *complex* coefficients is

$$\tilde{\alpha} \colon H^{2n-p}(X, \mathbb{C}) \to H^q(Y, \mathbb{C}), \qquad \eta \mapsto \langle \eta, \beta^{k,\ell} \rangle_X \cdot \gamma^{k',\ell'}.$$

Now, we use the Hodge–Riemann bilinear relations to conclude. The other direction is similar. □

This lemma allows us to construct interesting Hodge classes, by constructing appropriate morphisms of Hodge structures. To start, let X be an n-dimensional variety, and pick an integer $0 \leq q \leq n$. We have the identity

$$\mathrm{id} \in \operatorname{Hom}(H^q(X, \mathbb{Q}), H^q(X, \mathbb{Q})) \subset H^{2n}(X \times X, \mathbb{Q})$$

by using (2). this is clearly a morphism of Hodge structures of bidegree $(0, 0)$, called the *kth Künneth component of the diagonal*. Therefore, the associated element $\delta_q \in H^{2n}(X \times X, \mathbb{Q})$ is a Hodge class.

Conjecture 18 (Standard conjecture C, cf. [Voi14, 2.27])**.** *The classes δ_q are algebraic.*

There is a second set of operators that we can look at. Given an ample line bundle $\mathscr{L}$, we discussed before that the maps

$$L^{n-k} = - \cup (c_1(\mathscr{L}))^{n-k} \colon H^k(X, \mathbb{Q}) \to H^{2n-k}(X, \mathbb{Q})$$

are isomorphisms of Hodge structures. In particular, their *inverse* are also isomorphisms of Hodge structures, and so they define Hodge classes $\lambda_{n-k} \in H^{2k}(X, \mathbb{Q})$.

Conjecture 19 (Standard conjecture B, cf. [Voi14, 2.28])**.** *The classes λ_{n-k} are algebraic.*

At last, the third conjecture we will discuss today is not part of the *standard* conjectures, but has a similar flavor to the previous ones. (At least in this form, this is a conjecture of Voisin.) Informally, it says that we can take algebraic classes in the "correct" support.

Conjecture 20 (Voisin, 2013, cf. [Voi14, 2.29])**.** *Let X be a smooth complex algebraic variety, and let $Y \subset X$ be a closed algebraic subset. Let Z be a codimension k algebraic cycle on X, and assume that $[Z] \in H^{2k}(X, \mathbb{Q})$ restricts to zero in $H^{2k}(X - Y, \mathbb{Q})$. Then there exists a codimension k cycle Z' on X (with rational coefficients), supported on Y, such that $[Z] = [Z']$ in $H^{2k}(X, \mathbb{Q})$.*

Let us briefly discuss a simple case of this last conjecture.

Lemma 21 ([Voi14, 2.31])**.** *In the previous setup, we have that the conjecture holds if Y has codimension $\geq k - 1$. In particular, the conjecture holds for cycles of codimension 2.*

We will prove this conjecture only when Y is smooth. The only reason for this is that we will use the Hodge structure on $H^*(Y, \mathbb{Q})$. This can be replaced without difficulty with the *mixed Hodge structure* on Y, as we will discuss next week.

Proof. (Sketch) By assumption, the class $[Z]$ lies in the image of the map

$$H_{2n-2k}(Y, \mathbb{Q}) \to H_{2n-2k}(X, \mathbb{Q}),$$

where $n = \dim X$. Now, note that both sides carry Hodge structures (by Poincaré duality), and the map is a (bigraded) morphism of Hodge structures. By Corollary 9, we get that $[Z]$ is in the image of the Hodge classes of $H_{2n-2k}(Y, \mathbb{Q})$.

To conclude, we use our assumption on Y. We have that $2n - 2k$ is codimension 0 or 2 in Y. This way, all Hodge classes here are algebraic, as required. □

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