Cohomology theories

Q: What is a cohomology theory of sm. proj. varieties?

(1) $\forall X$ sm. proj., $H^\bullet(X)$ is a graded-commutative F-algebra. 08.02.24

(2) $f: X \to Y$ morphism $\rightsquigarrow$ $f^*: H^i(Y) \to H^i(X)$ pull-back

(3) $\gamma: CH^i(X) \to H^{2i}(X)$ homomorphism

(4) $\dim X = d$: $\int_X : H^{2d}(X) \to F$

satisfying a number of axioms, e.g.

- $H^i(X) = 0$ unless $i \in [0, 2d]$
- $\int_X : H^{2d}(X) \to F$ isom. if X is connected, $\dim(X) = d$.
- $H^\bullet(X) \otimes_F H^\bullet(Y) \xrightarrow{\sim} H^\bullet(X \times Y)$

The cycle class map

$X =$ sm. proj. var. $/ \mathbb{C}$, $\dim(X) = n$

$Z \overset{i}{\subseteq} X$ integral subscheme, $\dim(Z) = n-k$ 06.02.24

$\pi : \tilde{Z} \to Z$ resolution of singularities

Fact: $H_{2n-2k}(\tilde{Z}^{an}, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$ canonically via orientat. of $\tilde{Z}^{an}$

$PD : H_{2n-2k}(X^{an}, \mathbb{Z}) \xrightarrow{\sim} H^{2k}(X^{an}, \mathbb{Z})$ Poincaré duality

$\rightsquigarrow$ We define

$$[Z] := PD(i_* \underbrace{\pi_* 1}_{\in H_{2n-2k}(\tilde{Z}^{an}, \mathbb{Z})}) \in H^{2k}(X^{an}, \mathbb{Z})$$

Lemma[1]: $[Z] \in H^{2k}(X^{an}, \mathbb{Z})$ is independent of the choice of $\pi : \tilde{Z} \to Z$. (Exercise) (in algebraic topology)

$\Longrightarrow \widetilde{cl} : Z^k(X) \longrightarrow H^{2k}(X^{an}, \mathbb{Z})$

Var.– $V \subseteq X \times \mathbb{P}^1$, $f: V \to \mathbb{P}^1$, $p: X\times\mathbb{P}^1 \to \mathbb{P}^1$

Lemma[2]: If $\alpha \sim_{rat} 0$, then $cl(\alpha) = 0$. ~~(difficult exercise)~~ $f^*\widetilde{cl}(t) \cup \widetilde{cl}(V) = \widetilde{cl}(V_t)$

$\Longrightarrow cl : CH^k(X) \longrightarrow H^{2k}(X^{an}, \mathbb{Z})$ cycle class map.

Prop[3]: (a) $f : X \to Y$ morphism betwen sm. proj. var.

$\alpha \in CH^k(Y)$: $f^* cl(\alpha) = cl(f^*\alpha) \in H^{2k}(X^{an}, \mathbb{Z})$

(b) $f : X \to Y$ proper, $\alpha \in CH^k(X)$

$\Rightarrow f_* cl(\alpha) = cl(f_*\alpha) \in H^{2k+2r}(Y^{an}, \mathbb{Z})$,

$r = \dim(X) - \dim(Y)$

(c) X, Y sm. proj., $\alpha \in CH^k(X)$, $\beta \in CH^l(Y)$

$\Rightarrow cl(\alpha \times \beta) = cl(\alpha) \times cl(\beta) \in H^{2k}(X^{an}, \mathbb{Z}) \otimes H^{2l}(Y^{an}, \mathbb{Z})$.

$\downarrow$

$H^{2k+2l}(X^{an} \times Y^{an}, \mathbb{Z})$

07.02.24 Cor.[4] (d) $cl(\alpha \cdot \beta) = cl(\alpha) \cup cl(\beta)$

(e) $\Gamma \in CH^r(X \times Y)$ correspondence

$$\begin{array}{ccc} & X \times Y & \\ \swarrow^{p} & & \searrow^{q} \\ X & & Y \end{array}$$

$cl(\Gamma_* \alpha) = cl(q_*(p^*\alpha \cdot \Gamma))$

$\alpha \in CH^k(X)$ $= q_*(p^* cl(\alpha) \cup cl(\Gamma))$

$=: [\Gamma]_*(\alpha)$

Variant: 5 X is not projective but only quasi-projective

$X \subset \overline{X}$ compactification, $Z \subset X$ subvar.

~~$\rightsquigarrow \overline{Z} \subset \overline{X}$ compactification~~

$$CH^k(\overline{X}) \longrightarrow CH^k(X) \longrightarrow 0 \quad \text{surjective}$$

$\exists\, \overline{Z} \in CH^k(\overline{X})$: $\quad \overline{Z}|_X = Z$

$[Z] := [\overline{Z}]|_X \in H^{2k}(X^{an}, \mathbb{Z})$

Fact: This is indep. of the choice of $\overline{Z} \in CH^k(\overline{X})$ (easy exercise), and of the choice of $\overline{X} \supset X$. Prop. 3 holds for quasi-projective varieties as well.

Fact: $D \in A^1(X) \cong CH^1(X)$ Cartier divisor

$$\longrightarrow cl(D) = c_1(D) \in H^2(X^{an}, \mathbb{Z})$$

Example: (1) $X = \mathbb{P}^1$:

$$\begin{array}{ccc} CH^0(\mathbb{P}^1) & \longrightarrow & H^0(\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z} \\ \text{\rotatebox{90}{$\cong$}} \quad [\mathbb{P}^1] & \longmapsto & 1 \\ \mathbb{Z} & & \end{array}$$

$$\mathbb{Z} \cong CH^1(\mathbb{P}^1) \longrightarrow H^2(\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z}$$
$$[H] \longmapsto 1$$

(2) $X = E$ elliptic curve

$$\mathbb{Z} \cong CH^0(E) \longrightarrow H^0(E, \mathbb{Z}) \cong \mathbb{Z}$$

$$\mathrm{Pic}(E) \cong CH^1(E) \longrightarrow H^2(E, \mathbb{Z}) \cong \mathbb{Z}$$
$$D \longmapsto \deg D$$

Pure Chow motives

$X, Y/k$ sm. proj., possibly neither connected nor equidimensional

07.02.24

r-correspondences:

$$\mathrm{Corr}^r(X,Y)_{\mathbb{Q}} := \bigoplus_{d,e} CH^{d+r}(\underbrace{X_d \times Y_e}_{\text{equidim. parts}}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Category of pure Chow motives: $\mathrm{Chow}(k)$

objects: (X,p,m) with

- X/k sm. proj.
- $p \in \mathrm{Corr}^0(X,X)_{\mathbb{Q}}$ s.t. $p \circ p = p$
- $m \in \mathbb{Z}$.

(X,p,m) is called **effective** if $m=0$.

morphisms:

$$\mathrm{Hom}_{\mathrm{Chow}(k)}((X,p,m),(Y,q,m'))$$

$$:= \{ \underbrace{q}_{\mathrm{Corr}^0(Y,Y)_{\mathbb{Q}}} \circ \underbrace{\gamma}_{\mathrm{Corr}^{m'-m}(X,Y)_{\mathbb{Q}}} \circ \underbrace{p}_{\in \mathrm{Corr}^0(X,X)_{\mathbb{Q}}} \mid \gamma \in \mathrm{Corr}^{m'-m}(X,Y)_{\mathbb{Q}} \}$$

$(X,p,m) \in \mathrm{Chow}(k)$

$\rightsquigarrow$ $id = p \circ p \circ p = p$ identity morphism (since $p \circ p = p$)

Ex: X/k sm. proj., $\Delta_X \in \mathrm{Corr}^0(X,X)_{\mathbb{Q}}$ diagonal

$[X] := (X, \Delta_X, 0)$ motive of X

Ex: $\mathbb{L} = (\mathbb{P}^1, \mathbb{P}^1 \times \{0\}, 0)$ Lefschetz motive

with $p = [\mathbb{P}^1 \times \{0\}] \in CH^1(\mathbb{P}^1 \times \mathbb{P}^1)$

Exer: Show that $\mathbb{L}$ and $[\mathbb{P}^1] = (\mathbb{P}^1, \Delta_{\mathbb{P}^1}, 0)$ are not isomorphic.

Sol: $CH^1(\mathbb{P}^1 \times \mathbb{P}^1)_{\mathbb{Q}} = \mathbb{Q} \oplus \mathbb{Q}$ with $(a,b) \circ (c,d) = (ac, bd)$

$\mathrm{Hom}([\mathbb{P}^1], \mathbb{L}) = \{(0,a)\}$ $\quad$ ($[\mathbb{P}^1 \times \{0\}] = (0,1)$,

$\mathrm{Hom}(\mathbb{L}, [\mathbb{P}^1]) = \{(0,b)\}$ $\quad$ $\Delta_X = (1,1)$)

$(0,b) \circ (0,a) \neq (1,1) = id_{[\mathbb{P}^1]}$ $\quad \square$

Fact: Chow(k) is additive and $\mathbb{Q}$-linear.

$$(X,p,0)\oplus(Y,q,0) = (X\sqcup Y, p+q, 0)$$

Fact: Chow(k) is symmetric monoidal with tensor product

$$(X,p,m)\otimes(Y,q,m') := (X\times Y, p\times q, m+m').$$

The unit is $\mathbb{E} = (\phi, 0, 0)$.

Tate twist: $M = (X,p,m) \rightsquigarrow M(n) := (X,p,m+n)$.

Ex.: $\mathbb{L} \cong \mathbb{E}(-1)$

$\mathbb{L} \to \mathbb{E}(-1)$: $[\mathbb{P}^1] \in CH^0(\mathbb{P}^1 \times \{*\})_\mathbb{Q}$

$\mathbb{E}(-1) \to \mathbb{L}$: $[\mathcal{O}(1)] \in CH^1(\{*\} \times \mathbb{P}^1)_\mathbb{Q}$

Motives of different dimension can be isomorphic!

Ex.: $\mathbb{L}^t := (\mathbb{P}^1, \{0\}\times\mathbb{P}^1, 0)$

$\Rightarrow \mathbb{L}^t \not\cong \mathbb{E}$.

$\rightsquigarrow$ asymmetry

~~Moto~~ "Pure motives are obtained by formally inverting the Lefschetz motive."

Rem.: Weil cohomology theories are equivalent to $\mathbb{Q}$-linear symmetric monoidal functors (F/$\mathbb{Q}$ field ext.)

$$\sigma : \mathrm{Chow}(k) \longrightarrow \{\text{graded } F\text{-v.s.}\}$$

such that $\sigma(\mathbb{E}(1))$ lives in degree -2.

Ex: $\phi: X \to Y$ generically finite morphism of sm. proj. v. of dim. n

$$\Gamma_Y := \frac{(\phi,\phi)^*}{\deg\phi}\Delta_Y \in CH^n(X\times X)_\mathbb{Q}.$$

Fact: (1) $\Gamma_Y \circ \Gamma_Y = \Gamma_Y \rightsquigarrow (X, \Gamma_Y, 0) \in \mathrm{Chow}(k)$

(2) $\Gamma_\phi \in CH^n(X\times Y)_\mathbb{Q}$ defines an isomorphism $(X,\Gamma_Y,0) \xrightarrow{\sim} (Y, \Delta_Y, 0)$.

graph

Ex.: X sm. proj., $\dim X = n$, $G \circlearrowright X$ action of finite group

$g \in G \rightsquigarrow$ graph $\Gamma_g \in CH^n(X\times X)_\mathbb{Q}$

$\chi: G \longrightarrow \{\pm 1\}$ character (= group homomorphism)

$$p_\chi = \frac{1}{|G|}\sum_{g\in G}\chi(g)\,\Gamma_g \in CH^n(X\times X)_\mathbb{Q}$$

$\Gamma_g \circ \Gamma_{g'} = \Gamma_{gg'} \rightsquigarrow p_\chi \circ p_\chi = p_\chi$ (straightforward) $\rightsquigarrow$ motive $(X, p_\chi, 0)$.