Chow groups

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§1. Definition of Chow group

k: field. X: var/k (= integral separated scheme of fin type/k)

Define a free abelian group

$$Z_k(X) := \mathbb{Z}\{W \subset X \text{ closed subvar of dim } k\}$$

Any elem. in $Z_k(X)$ is called an algebraic k-cycle of X.

<u>Recall</u> Say $\dim X = n$. Then $Z_{n-1}(X)$ is nothing but the Weil divisor group.

If X: normal then for each $f \in K^* = K(X) \setminus 0$, we can define

$$\operatorname{div} f := \sum_{D \subset X} \nu_D(f) \cdot D \in Z_{n-1}(X).$$

This defines $\mathrm{div}: K^* \longrightarrow Z_{n-1}(X)$ gp hom.

Consider its image $Z_{n-1}(X)_{rat} \subset Z_{n-1}(X)$. The quotient $Z_{n-1}/Z_{n-1,rat} =: CH_{n-1}(X)$ is the Weil class group. If X: regular then this is the Picard group.

Now let's generalize this to lower dim'l subvarieties. Say $V \subset X$: $(k+1)$-dim'l subvar. Consider its normalization $\tilde{V} \to V \hookrightarrow X$. Since $\tilde{V}$: normal, we can define $Z_k(\tilde{V})_{rat} \subset Z_k(\tilde{V})$.

In general, for any <u>proper</u> map $f: X \to Y$ we can define a gp hom
$f_*: Z_k(X) \longrightarrow Z_k(Y)$, where $f_*(W)$ is defined as follows:

Consider the image $f(W)$, a closed subvar of Y. If $\dim f(W) < k$ (ie, f contracts W) then we set $f_* W := 0$. If $\dim f(W) = k$ (ie f is gen. fin. on W) then set

$$f_* W := [K(W) : K(f(W))] \cdot f(W) \in Z_k(Y).$$

Apply this pushforward hom to $\tilde{V} \xrightarrow{\text{fin}} V \xhookrightarrow{\text{cl}} X$

$$\Rightarrow Z_k(\tilde{V}) \longrightarrow Z_k(X).$$

<u>Def</u> $$Z_k(X)_{rat} := \sum_{\substack{V \subset X \\ \dim k+1}} \operatorname{im}\left(Z_k(\tilde{V})_{rat} \longrightarrow Z_k(X)\right)$$

Define the Chow group by

$$CH_k(X) := Z_k(X) / Z_k(X)_{rat} : \text{abelian group.}$$

This generalizes the Weil class gp $CH_{n-1}(X)$.

Rmk In fact, this defn works for any separated scheme (alg. space) of fin. type / k.

Ex X : sep scheme fin. type / k. $\rightsquigarrow CH_*(X)$.

Say $X = \bigcup_{i=1}^{t} X_i$ irr. decomp. Define the (fundamental) cycle of X by

$$[X] := \sum_{i=1}^{t} \mathrm{length}(\mathcal{O}_{X,\xi_i}) \cdot [X_i] \in Z_*(X).$$

$$\downarrow$$

$$CH_*(X).$$

If $Z \overset{i}{\subset} X$ closed subscheme, define $[Z] \in CH_*(X)$ by the image of the fund. cycle of Z via $i_* : Z_*(Z) \longrightarrow Z_*(X) \twoheadrightarrow CH_*(X)$.

<u>Ex</u> X; var/k of dim n. Then by defn,

$CH_n(X) = Z_n(X) = \mathbb{Z}\cdot[X]$.

<u>Ex</u> C; sm proj curve / $\mathbb{C}$ w/ genus g. There are $CH_0(C)$ and $CH_1(C) = \mathbb{Z}\cdot[C]$. Since C: smooth, $CH_0(C) = \mathrm{Pic}(C)$.

<u>Fact</u> $0 \to \mathrm{Pic}^\circ(C) \to \mathrm{Pic}\,C \xrightarrow{\deg} \mathbb{Z} \to 0$

and $\mathrm{Pic}^\circ(C) = J(C)$, where J is an abelian variety of dimension g.

Therefore,

- $CH_0(\mathbb{P}^1) \cong \mathbb{Z}$ (by degree of 0-cycles)
- $CH_0(C)$ is already huge if $g \geq 1$.

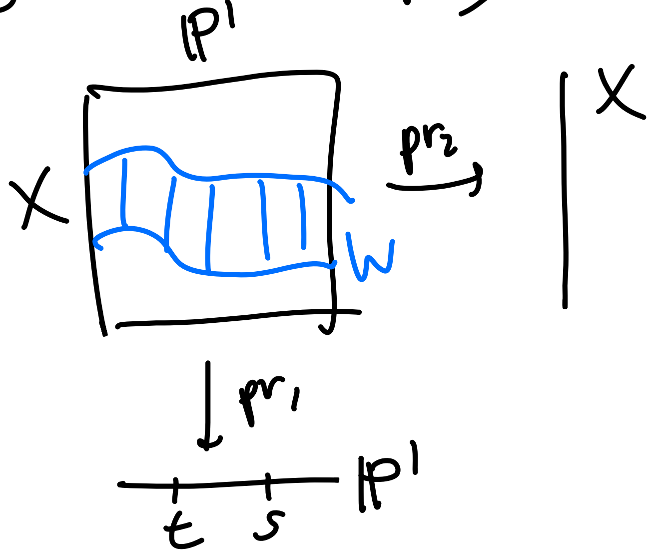
Ex Similarly, if X: sm proj$/\mathbb{C}$ of dim n then

$CH_{n-1}(X) = \mathrm{Pic}(X)$, $\quad 0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}\,X \xrightarrow{c_1} NS(X) \to 0$

$\mathrm{Pic}^0(X) = \mathrm{Pic}^0_X(\mathbb{C})$: ab. var dim $h^{1,0}(X)$.

$\therefore CH_{n-1}(X)$ is huge iff $b_1(X) > 0$.

Rmk (Alternative defn) The rat'l equiv. is an equiv. on k-cycles where $\alpha \sim_{rat} \beta$ if one can deform α to β along a sequence of rat'l curves. More specifically, if we consider a flat deformation of k-dim'l cycles $W \to \mathbb{P}^1$, then we set $W_s \sim_{rat} W_t$ for $\forall s,t$.



Rat'l equiv. is generated by such equivalence.

$$\left(\begin{array}{l}\text{ie, for any } f \in k(\mathbb{P}^1), \text{ set } (p_2)_* p_1^*(\operatorname{div} f) = 0 \\ \mathbb{Z} = CH_0(\mathbb{P}^1) \xrightarrow{p_1^*} CH_k(W) \xrightarrow{(p_2)_*} CH_k(X) \\ \quad [\operatorname{div} f] = 0 \longmapsto 0 \end{array}\right)$$

§2. Push forward, pullback, $\mathbb{A}^1$-invariance, localizations, and intersections

Given $f: X \to Y$ proper morphism, we have already defined $f_*: Z_k(X) \to Z_k(Y)$.

<u>Prop</u> $f: X \to Y$ proper $\Rightarrow$ $f_*: CH_k(X) \to CH_k(Y)$

<u>pf</u> ETS: f_* sends $Z_k(X)_{rat}$ to $Z_k(Y)_{rat}$.

Say
$$\begin{tikzcd} \tilde{V} \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "f"] \\ \tilde{W} \arrow[r] & Y \end{tikzcd}$$
, where $W = f(V)$ with $\dim V = \dim W = k+1$.

Since $K(V)/K(W)$ is finite, $\exists$ norm homomorphism $N: K(V)^* \to K(W)^*$.

<u>Fact</u> $g_*(\operatorname{div} h) = \operatorname{div}(N(h))$ for $h \in K(V)^*$.

Now use commutativity. $\blacksquare$

Say $f: X \to Y$ is flat of rel. dim l.

Then any $W \subset Y$ closed subvar. of dim k s.t. $f(X) \cap W \neq \emptyset \Rightarrow f^{-1}W \subset X$ closed subvar of dim $k+l$.

This defines $f^*: Z_k(Y) \to Z_{k+l}(X)$.

<u>Prop</u> $f: X \to Y$ flat $\Rightarrow f^*: CH_k(Y) \to CH_{k+l}(X)$.

<u>pf</u> (sketch) Any $\mathbb{P}^1$-family of k-dim'l cycles on Y pull backs to a $\mathbb{P}^1$-family of $(k+l)$-dim'l cycles on X. Use the alternative defn. $\blacksquare$

Even when f is not flat, if Y is regular then we can define f^* by the following trick:

① Factorize

$$\begin{tikzcd} x \arrow[r, mapsto] & (x, f(x)) \\ X \arrow[r, hook, "i"] \arrow[dr, "f"'] & X \times Y \arrow[d, "pr_2"] \\ & Y \end{tikzcd}$$

i: regular immersion since Y: regular.

(if $a_1, \dots, a_m \in \mathcal{O}_Y$ is a regular sequence

then $a_i - f^\#(a_i) \in \mathcal{O}_{X \times Y}$ defines $i(X)$)

② Since pr_2 is flat, $\exists\, pr_2^*$.

Since i: regular immersion, define

i^* separately for this case.

$\Rightarrow f^* := pr_2^* \circ i^*$

Basically refined intersection product. Nontrivial.

Thm X: regular of dim n. Then $\exists$ an intersection pairing $CH_k(X) \otimes CH_\ell(X) \to CH_{k+\ell-n}(X)$.

If $W, V \subset X$ are k-dim'l (resp. ℓ-dim'l) subvar s.t. $\dim(W \cap V) = k+\ell-n$ (W and V are intersecting properly) then

$$[W]\cdot[V] = \sum_{Z_i \subseteq W \cap V} m_i \cdot [Z_i] \quad \text{for } m_i \in \mathbb{Z}_{>0}$$

(called intersection multiplicity)

pf $W, V \subset X$ subvars.

$\Rightarrow$ $W \times V \subset X^2$ is a subvar. dim $k+\ell$.

$\Rightarrow$ $[W]\cdot[V] := \Delta^*[W \times V]$ for $\Delta : X \hookrightarrow X^2$.

Here Δ^* is well-defn since Δ is a reg. imm. ∎

Cor If X: regular of dim n then $CH^*(X)$ w/ int. pairing is a ring ($CH^k(X) := CH_{n-k}(X)$)

Now $\mathbb{A}^1$-invariance:

<u>Prop</u> Let $p: \mathbb{A}^1 \times X \to X$ be the 2^{nd} proj.

Then $p^*: CH_k(X) \longrightarrow CH_{k+1}(\mathbb{A}^1 \times X)$ is an isom.

<u>pf (idea)</u> $\exists$ section $e: X \longrightarrow \mathbb{A}^1 \times X$ of p.

$\Rightarrow CH_*(X) \xrightarrow{p^*} CH_{*+1}(\mathbb{A}^1 \times X) \xrightarrow{e^*} CH_*(X)$ is id.

$\therefore$ p^* is injective.

e: regular immersion.

Surjectivity is harder. Let's do it only for $k = n-1$. Assume $W \subset \mathbb{A}^1 \times X$ has dim n.

If $W = p^{-1}(V)$ then all good. Otherwise, it dominates $X \Rightarrow$ Over generic pt, defines $W_K \subset \mathbb{A}^1_{K(X)}$

$\Rightarrow$ Cut out by $f \in K(X)[t] \subset K(\mathbb{A}^1 \times X)$

$\therefore$ $W = \operatorname{div} f$ generically, ie, $W - \operatorname{div} f$ is supp. on div of X. $\blacksquare$

Localization sequence:

<u>Prop</u> $Y \overset{i}{\hookrightarrow} X$ closed subscheme and $U := X \setminus Y \overset{j}{\hookrightarrow} X$.

Then $\exists$ right exact seq.

$$CH_k(Y) \xrightarrow{i_*} CH_k(X) \xrightarrow{j^*} CH_k(U) \to 0.$$

<u>pf</u> $j^* \circ i_* = 0$ is clear from defn.

If $\alpha \in CH_k(X)$ and $j^*\alpha = 0 \in CH_k(U)$,

then $j^*\alpha = \underset{\substack{\cap \\ Z_k(U)}}{\sum_i \operatorname{div} f_i}$ for $f_i \in K(W_i)$ ($W_i \subset U$ dim $k+1$)

Take the Zar. closure $\overline{W_i} \subset X$.

$\Rightarrow j^*(\alpha - \sum \operatorname{div} f_i) = 0 \in Z_k(U)$ for $f_i \in K(\overline{W_i})$

$\Rightarrow \alpha - \sum \operatorname{div} f_i \in \operatorname{im} i_*$ by defn of j^*. $\boxtimes$

§3. More examples

Ex $CH_k(\mathbb{A}^n) = \begin{cases} \mathbb{Z} & \text{if } k=n \\ 0 & \text{otherwise} \end{cases}$

(Use $\mathbb{A}^1$-invariance)

$CH_k(\mathbb{P}^n) = \mathbb{Z}$ for $0 \le k \le n$.

(Use localization sequence)

Ex (Mumford's theorem) S: K3 surf.

Then $CH_0(S)$ should be very large.

Mumford: $\forall C \overset{i}{\hookrightarrow} S$ any 1-dim'l closed subscheme, $CH_0(C) \xrightarrow{i_*} CH_0(S)$ cannot be surjective.

§4. Correspondence

<u>Def</u> X, Y: smooth varieties. We define

$$\mathrm{Corr}(X,Y) := CH^{\dim X}(X \times Y)$$ and call its element by a (degree 0) correspondence from X to Y.

<u>Ex</u> Let $f: Y \to X$ morphism. Then the image of $Y \hookrightarrow X \times Y$, $f \mapsto (f(y), y)$ defines $[\Gamma] \in CH^{\dim X}(X \times Y) = \mathrm{Corr}(X,Y)$.

<u>Def</u> Let X, Y, Z: sm proj $/k$. If $\alpha \in \mathrm{Corr}(X,Y)$, $\beta \in \mathrm{Corr}(Y,Z)$ then we define

$$\beta \circ \alpha := (pr_{XZ})_* (pr_{XY}^* \alpha \cdot pr_{YZ}^* \beta) \in \mathrm{Corr}(X,Z).$$

<u>Prop</u> $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$.

<u>pf</u> Projection formula. $\blacksquare$

This is going to be a "Hom-set" between X & Y considered as "motives".