INTRO TALK: CHOW RINGS AND DECOMPOSITION OF THE DIAGONAL

NATHAN CHEN
JANUARY 26, 2024

Consider $\pi\colon \mathcal{X} \to B$ a smooth projective morphism. Here, B is smooth irreducible quasi-projective. Let $Z \subset \mathcal{X}$ be a codim k cycle.

Fundamental fact/observation:

Theorem 1. *If $Z_t \sim_{rat} 0$ (for some $t \in B$), then there exists a Zariski open set $U \subset B$ and a nonzero integer N such that*

$$N \cdot Z\big|_{\mathcal{X}_U} \sim_{rat} 0.$$

Note: by a Baire category argument, this is actually equivalent to the assumption that Z_t is rationally equivalent to 0 for a very general point $t \in B$.

This is known as the spreading-out phenomenon for rational equivalence.

Corollary 2. *In the above, there exists $U \subset B$ dense Zariski-open such that the Betti cycle class*

$$[Z] \in H^{2k}_B(\mathcal{X}, \mathbb{Q})$$

vanishes on the open set $\mathcal{X}_U$.

One can actually apply this to the case where the family $\mathcal{X} \to B$ is a product $X \times B$. This gives decomposition principle due to Bloch and Srinivas. Rephrase theorem above... let $Z \subset X \times B$ and consider the map

$$\mathrm{CH}_0(B) \to \mathrm{CH}^k(X), \qquad b \mapsto Z_b.$$

If this map is the zero map, then the cycle Z vanishes up to torsion on some open set of the form $X \times U$ with U as above.

Set-up: consider $X = Y \setminus W$ where Y = smooth projective, $W \subset Y$ closed algebraic subset. Let $B = Y$. Let Z = restriction of Δ_Y to $(Y \setminus W) \times Y$. Then the map

$$\mathrm{CH}_0(B) \to \mathrm{CH}_0(X), \qquad b \mapsto Z_b$$

is trivial (i.e. the zero map) iff (by localization exact seq.) every point of Y is rationally equivalent to 0-cycle supported on W.

Conclusion: $\Delta\big|_{(Y \setminus W) \times U}$ is torsion. By localization, this says that

$$N \cdot \Delta_Y \sim_{rat, Y \times Y} \underbrace{(\quad\quad)}_{\text{supp on } W \times Y} + \underbrace{(\quad\quad)}_{\text{supp on } Y \times D}$$

for some N, where $D = Y \setminus U$.

Similar version in cohomology.

Application (Bloch-Srinivas)

Theorem 3. *X smooth projective, $W \subset X$ closed alg. subset of dim $\leq k$ such that any point of X is rationally equivalent to 0-cycle supp on W. Then*

$$H^0(X, \Omega_X^\ell) = 0$$

for $\ell > k$.

Further on, we'll see *coniveau*. A Betti cohomology class has geometric coniveau $\geq c$ if supported on a closed algebraic subset of codim $\geq c$. *Hodge coniveau* computed by looking at shape of Hodge structures on $H_B^*(X, \mathbb{Q})$.

The generalized Hodge conjecture (Grothendieck) identifies geometric coniveau to the Hodge coniveau.

Idea goes back to Mumford, who shows that on a surface S, there is a strong relationship between $\mathrm{CH}_0(S)$ and $H^0(\Omega_S^1)$.

Theorem 4. *If $H^{2,0}(S) \neq 0$, then no curve $j\colon C \hookrightarrow S$ satisfies the property that $j_*\colon \mathrm{CH}_0(C) \to \mathrm{CH}_0(S)$ is surjective.*

Rephrase Mumford's theorem as saying that if the degree 2 cohomology of S is not supported on any divisor (related to $H^{2,0}(S) \neq 0$ by the Lefschetz $(1,1)$-theorem), then its Chow group $\mathrm{CH}_0(S)$ is not supported on any divisor.

We'll then cover some of the statements surrounding generalized Bloch's conjecture: if the transcendental cohomology $H_B^*(X, \mathbb{Q})^{\perp \mathrm{alg}}$ has coniveau $\geq c$, then the cycle class map

$$\mathrm{cl}\colon CH_0(X, \mathbb{Q}) \to H_B^{2n-2i}(X, \mathbb{Q})$$

is injective for $i \leq c - 1$.

More applications of decomposition of the diagonal: Mumford-Roitman theorem, which is the converse to the generalized Bloch's conjecture.

Talks (aim for 1 hour):

(1) February 2. Start of Chapter 2 on Chow groups, functoriality. Stop at motives.

(2) February 9. Motives (2.1.3.1), Cycle class (2.1.4).

(3) February 16. Hodge structures (2.2.1 - 2.2.3), standard conjectures.

(4) February 23. Mixed HS (2.2.4) + Coniveau (2.2.5).

(5) March 1. General principles (3.1) + Mumford's theorem (3.1.1). Maybe sketch a proof of decomposition of diagonal?

(6) March 8.... continue?

(7) March 15. Spring break.

(8) March 22. Further applications - three theorems of Bloch-Srinivas (3.1.2).

(9) March 29. Generalized decomposition of diagonal (3.2.1)

(10) April 5. Generalized Bloch conjecture (3.2.2)

(11) April 12. Kimura + Nilpotence theorem: completely different approach to Bloch, which works for varieties dominated by projects of curves.

(12) April 19. Hodge coniveau of complete intersections?

(13) April 26...