NOTE ON OPENAI'S PROOF OF THE BASE OF LAGRANGIAN FIBRATIONS

YOON-JOO KIM

Abstract. OpenAI [Open26] proved that every Lagrangian fibration of a compact hyper-Kähler manifold has a base $\mathbb{P}^n$. This expository note summarizes the last section of their paper.

Disclosures. I declare that all work here is my human effort on understanding the paper [Open26]. No AI was used in this process, neither to understand the paper nor to write up this note (except that [Open26] was entirely written by AI). All the results here are either explicitly or tacitly contained in OpenAI's paper, and I claim no originality. Consider this as an expository seminar note.

Shortly after OpenAI's paper has appeared, Zhiyu Liu announced a separate paper [Liu26]. Their Theorem 1.1 extracted an optimal statement from OpenAI's argument. I will follow Liu's formulation, but will exclusively follow OpenAI's paper for the body of the note.

1. Introduction

The goal of this note is to understand the proof of [Open26, Proposition 11.1]. We will reference this paper many times throughout the note, so we will simply write AI::Proposition 11.1 instead of citing the paper every time.

Theorem 1.1 (AI::Proposition 11.1)**.** *Let $f : X \to B$ be a Lagrangian fibration of a projective hyper-Kähler manifold X of dimension $2n$. Suppose that B has quotient singularities and there exists a finite surjective morphism $p : A \to B$ from an abelian variety A. Then B is smooth.*

This theorem is merely the last step of OpenAI's main result, where the singularity assumption on B and the existence of p was completely dropped. More than half of the paper is devoted to proving B has quotient singularities, and AI::Proposition 9.1 in the paper shows such a morphism $p : A \to B$ always exists. We will not discuss these parts at all, and will only focus on proving their AI::Proposition 11.1 (Theorem 1.1 in this note).

The first step of the proof is to take the canonical smooth DM stack $\mathfrak{B} \to B$ and lift f and p to $\tilde{f} : X \to \mathfrak{B}$ and $\tilde{p} : A \to \mathfrak{B}$ (AI::Lemma 11.2) using [FMN10, Theorem 4.6]. Note that $\tilde{f}$ and $\tilde{p}$ are flat because $\mathfrak{B}$ is smooth. To ease notation, we will write B for the canonical smooth DM stack from now on, and write

$$f : X \to B, \qquad p : A \to B$$

for the new stacky-versions of these morphisms. The original base variety of the Lagrangian fibration in Theorem 1.1 will be denoted by B^{sch}, the coarse moduli space of B.

Theorem 1.1 rarely uses any input from hyper-Kähler manifolds or Lagrangian fibrations. In fact, a more technically precise version of OpenAI's result is the following, which was already successfully extracted in [Liu26, Theorem 1.1].

Date: November 15, 2026.

Theorem 1.2. *Let B be a smooth proper DM stack with trivial stabilizers in codimension 1, and admitting a finite flat morphism $p : A \to B$ from an abelian variety A. Assume*

(i) *The coarse moduli space B^{sch} is a Fano variety.*
(ii) *Every connected component of the inertia stack I_B, except the identity component, has the Euler characteristic 0.*

Then B is a smooth projective variety.

Here the Euler characteristic of a DM stack means a stacky-Euler characteristic that can be a rational number in general. The stacky Gauss–Bonnet theorem [Beh09, Proposition 1.6] shows that (ii) is equivalent to the vanishing of $c_{top}(T_{I_B})$ on each non-identity component of I_B. Hence Theorem 1.2 is identical to [Liu26, Theorem 1.1]. OpenAI's proof of this theorem is extremely systematic and geometric. However, the paper is difficult to grasp at first sight due to its terse presentation. Throughout the note, we will expand OpenAI's original proof and make its beautiful geometric ideas more explicit.

Here is a quick summary of the structure of this note. §2 reviews and introduces basic definitions we use throughout the note: three keywords are groupoid schemes, their inertial components, and the difference morphisms. The following two sections prove Theorem 3.1 assuming condition (i) in Theorem 1.2, and prove Theorem 4.1 from condition (ii). If B is not a variety then these two conclusions are incompatible, thereby completing the proof of the theorem. The last §5 completes the proof of the desired Theorem 1.1.

Acknowledgments. I thank Andres Fernandez Herrero, James Hotchkiss, and Niklas Müller for helpful comments on the draft of this note.

Notation and conventions. We work over a fixed algebraically closed field k of characteristic 0. A *variety* in this note is a reduced and separated scheme of finite type over k; it may be reducible or disconnected unless stated otherwise.

We mostly follow OpenAI's original notation, so that the reader can easily compare our exposition to the original reference. Some of the original notation looked a bit awkward to us, and in those cases, we tried to choose the most reasonable alternative.

2. Inertial components and the difference map

This section focuses on the abstract study of finite flat morphisms $p : A \to B$ from an abelian variety A to a smooth proper DM stack B (with no stabilizers in codimension 1). No additional assumptions will be imposed in this section.

We will define the three main objects of this note. First, we will recall the definition of the *groupoid scheme* $R \rightrightarrows A$ associated to the morphism p. Second, we will introduce the notion of its *inertial component* R_i (Definition 2.7). Third, we will define the *difference morphism* $\delta : R \to A$ (Definition 2.13). The proof of Theorem 1.2 in one sentence will be:

the study of inertial components $R_i \subset R$ under the difference morphism $\delta : R \to A$.

2.1. Finite flat groupoid schemes.

We will heavily use the theory of algebraic stacks, or more precisely the groupoid-quotient aspects of them. Our main reference is the chapters "Groupoid schemes" and "Groupoids in algebraic spaces" in [Stacks, Tag 022L, 0437], which in turn references the original paper [KM97].

Observe that the finite flat morphism $p : A \to B$ can be thought of as a finite flat presentation of the DM stack B. This allows one to view B as a flat quotient stack $B = [A/R]$ of the abelian variety A by a *groupoid scheme* R over it. The latter is defined as the fiber product

$$\begin{array}{ccc} R \coloneqq A \times_B A & \xrightarrow{t} & A \\ \downarrow s & & \downarrow p \\ A & \xrightarrow{p} & B \end{array}.$$

The two projection maps $s, t : R \to A$ are finite flat morphisms of schemes. Take the irreducible decomposition

$$R = \bigcup_i R_i \qquad \text{(with reduced induced structures)}.$$

The irreducible components R_i will play very important roles throughout.

Lemma 2.1. *R is a projective Cohen–Macaulay variety with pure dimension n. Moreover, for every irreducible component $R_i \subset R$, the restricted morphisms $s_{|R_i}, t_{|R_i} : R_i \to A$ are finite surjective.*

Proof. Since R is finite flat over A, it is a projective Cohen–Macaulay scheme and all its irreducible components R_i are finite surjective over A. Since $p : A \to B$ is generically étale, R satisfies the condition (R_0) and hence it is reduced. □

The variety R parametrizes triples (x, y, g) of the form

$$R = \{(x, y, g) : \ x, y \in A, \ \ p(x) = p(y) \in B, \ \ g \in \mathrm{Aut}(p(x))\}.$$

The forgetful map defines a finite morphism

$$\mathrm{forget} \coloneqq (s, t) : R \to A \times A, \qquad (x, y, g) \mapsto (x, y).$$

We denote the image of R in $A \times A$ by $\bar{R} = \mathrm{forget}(R)$, as well as $\bar{R}_i = \mathrm{forget}(R_i)$.

Lemma 2.2. *The forgetful morphism $R \to A \times A$ is finite and generically injective. In particular, $R_i \to \bar{R}_i$ is a finite birational morphism for each i, so they share the same normalization Z_i.*[1]

Proof. The DM stack B has a trivial generic stabilizer, so $R \to A \times A$ is generically injective on each irreducible component R_i. □

2.1.1. *Compositions in groupoid.* The groupoid R comes equipped with a composition morphism in its datum. More precisely, consider the fiber product

$$\begin{array}{ccc} R \times_A R & \xrightarrow{p_2} & R \\ \downarrow p_1 & & \downarrow s \\ R & \xrightarrow{t} & A \end{array}, \qquad \begin{aligned} R \times_A R &\coloneqq \{\Big((x, y, g), (y', z, h)\Big) \in R \times R : y = y'\} \\ &\cong A \times_B A \times_B A \end{aligned}.$$

Projection map to the first-third factors is the *composition* morphism

$$c : R \times_A R \to R, \qquad (x, y, z, g, h) \mapsto (x, z, h \circ g).$$

Lemma 2.3. *$R \times_A R$ is a projective Cohen–Macaulay variety, whose irreducible components are n-dimensional and finite surjective over A.*

Proof. Same as Lemma 2.1. □

Here is a slightly stronger version:

[1] The original paper often uses the normalization Z_i instead of R_i. We will try to stick to the original variety R_i as much as possible.

Lemma 2.4. *Let $R_i, R_j \subset R$ be irreducible components. Then the closed subscheme $R_i \times_A R_j \subset R \times_A R$ satisfies the following.*[2]

(1) *Every irreducible component $T \subset R_i \times_A R_j$ is an irreducible component of $R \times_A R$.*
(2) *$R_i = p_1(T)$, $R_j = p_2(T)$, and $R_k \coloneqq c(T)$ are all irreducible components of R.*
(3) *T is finite surjective over R_i, R_j, and R_k.*

Proof. (1) Notice that $R_i \times_A R_j$ is the preimage of $\Delta_A \subset A \times A$ under the morphism $(t, s) : R_i \times R_j \to A \times A$. Since $R_i \times R_j$ has pure dimension $2n$ and $\Delta_A \subset A \times A$ is a regular closed immersion of codimension n, we conclude that $R_i \times_A R_j$ has pure dimension n. Therefore, $R_i \times_A R_j$ is a union of irreducible components of $R \times_A R$.

(2) Note that c is finite, because both $R \times_A R$ and R are finite over A. Hence $c(T)$ has dimension n, meaning it is an irreducible component of R.

(3) All morphisms s, t, and c are finite, so T is finite over its images. □

2.2. **Inertial components.** The DM stack B has an inertia group stack $I_B \to B$. Its pullback

$$\begin{array}{ccc} I_A & \longrightarrow & I_B \\ \downarrow & & \downarrow \\ A & \xrightarrow{p} & B \end{array} \tag{2.5}$$

is a finite group scheme $I_A \to A$ encoding every inertial information of B.

Definition 2.6. The finite group scheme $I_A \to A$ in (2.5) is called the *stabilizer group scheme* (or *inertia group scheme*).[3]

Note that I_A parametrizes the pairs (x, g) where $x \in A$ and g is an automorphism of the point $p(x) \in B$. Note that I_A may be a non-reduced scheme. We will often consider I_A as a closed subscheme of R via $I_A = \text{forget}^{-1}(\Delta_A)$, or equivalently

$$I_A \hookrightarrow R, \qquad (x, g) \mapsto (x, x, g).$$

Under this identification, the structural morphism $I_A \to A$ is the restriction of the morphism $s : R \to A$ (or t)

$$s : I_A \to A, \qquad (x, x, g) \mapsto x.$$

Definition 2.7. (1) The *identity component* of R is the irreducible component

$$R_0 = \{(x, x, \text{id}) \in R\} \quad \subset \quad R$$

that corresponds to the identity section of the group scheme $I_A \to A$.
(2) A *non-identity component* of R is an irreducible component $R_i \subset R$ that is not R_0.
(3) An *inertial component* of R is an irreducible component $R_i \subset R$ containing a non-identity element of I_A, i.e., R_i contains a point of the form (x, x, g) with $g \neq \text{id}$.[4]

Lemma 2.8. (1) *The morphism $I_A \to A$ is unramified.*
(2) *Every connected component S of I_A satisfies either $S = R_0$ or $S \cap R_0 = \varnothing$.*

Proof. (1) Every finite group scheme in characteristic 0 is unramified by Cartier's theorem.

(2) Since $I_A \to A$ is unramified, étale locally over A, it is a closed immersion. Hence the identity component can R_0 never intersect with a non-identity component S. □

[2] This is tacitly proved in p.94 of the paper.
[3] The paper calls I_A the difference-zero set and did not assign it a symbol. See Lemma 2.15.
[4] The paper calls a point (x, x, g) with $g \neq \text{id}$ a "non-identity arrow of difference zero".

Remark 2.9. A non-identity component $R_i \subset R$ may very well satisfy $R_i \cap R_0 \neq \varnothing$, see Lemma 2.11. This will be crucial in Lemma 3.9.

Proposition 2.10. *The following are equivalent.*

(1) $B \to B^{sch}$ *is an isomorphism, i.e., B is an algebraic space.*
(2) forget : $R \to A \times A$ *is a closed immersion.*
(3) *The stabilizer group scheme $I_A \to A$ is trivial.*
(4) *R has no inertial components.*

Proof. The equivalence of (1–3) and the implication (3) $\Longrightarrow$ (4) are clear. For (4) $\Longrightarrow$ (3), assume that $I_A \to A$ is nontrivial, so that there exists an irreducible component $S(\neq R_0) \subset I_A$. Since $S \subset I_A \subset R$, there exists an irreducible component $R_i \subset R$ containing S. Notice that $S \cap R_0 = \varnothing$ by Lemma 2.8, so R_i is inertial. □

The following two lemmas describe the local behaviors of R around the points $(x, x, g) \in I_A \subset R$. These are completely formal.

Lemma 2.11. (1) *$R_0 \to \bar{R}_0 = \Delta_A$ is an isomorphism.*

(2) *Let $\eta \in A$ be a codimension 1 point. Then the point $(\eta, \eta, \mathrm{id}) \in R$ admits an étale local neighborhood W of the form*

$$W \subset \{(x, y) \in \mathbb{A}^2 : x^e = y^e\} \times \mathbb{A}^{n-1},$$

where the identity branch $R_0 \subset R$ corresponds to $\{(x, x)\} \times \mathbb{A}^{n-1} \subset W$.

Proof. (1) Clear. (2) Locally, $p : A \to B$ is a finite morphism between smooth varieties. Therefore, around a codimension 1 point $\eta \in A$ it has a form

$$p : \mathbb{A}^1_x \times \mathbb{A}^{n-1}_u \to \mathbb{A}^1_z \times \mathbb{A}^{n-1}_u, \qquad (x, u) \mapsto (x^e, u).$$

The fiber product $R = A \times_B A$ around $(\eta, \eta, \mathrm{id})$ therefore has the desired description. □

Lemma 2.12. *Every point of the form $(x_0, x_0, g_0) \in R$ with $g_0 \neq \mathrm{id}$ admits an étale local neighborhood W satisfying the following.*

(a) *There exists an étale neighborhood U of $x_0 \in A$, a neighborhood $[V/G]$ of $p(x_0) \in B$, and a morphism $f : U \to V$ such that p has a local presentation*

$$\bar{f} : U \xrightarrow{f} V \longrightarrow [V/G] \ .$$

(b) *There exists an étale neighborhood W of $(x_0, x_0, g_0) \in R_i$ such that every point (x, y, g_0) lying in W satisfies*

$$f(y) = g_0 \cdot f(x) \qquad \text{in } V.$$

Proof. The DM stack B admits a neighborhood $[V/G]$ around $p(x_0)$, where G is the automorphism group of $p(x_0)$ and V is a smooth affine variety with a G-action fixing $0 \in V$ corresponding to $p(x_0)$ [AV02, Lemma 2.2.3] [Ols06, Theorem 2.12]. Take any neighborhood U of $x \in A$ so that we have a restriction $\bar{f} := p_{|U} : U \to [V/G]$.

$$\begin{array}{ccccc} & & A & \xrightarrow{p} & B \\ & & {\scriptstyle ét}\uparrow & & \uparrow{\scriptstyle ét} \\ & & U & \xrightarrow{\bar{f}} & [V/G] \\ & {\scriptstyle \mathrm{id}}\nearrow & \uparrow & & \uparrow{\scriptstyle G} \\ U & \hookrightarrow & U \times G & \longrightarrow & V \end{array}$$

Shrinking U further, we may assume that $V \to [V/G]$ pullbacks to $U \times G \to U$. Taking a component $U \hookrightarrow U \times G$ by $u \mapsto (u, \mathrm{id})$, the third row is the desired morphism $f : U \to V$.[5]

The point $(x_0, x_0, g_0) \in R$ admits a neighborhood R_U given by

$$\begin{array}{ccc} R_U & \xrightarrow{t} & U \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \bar{f}} \\ U & \xrightarrow{\bar{f}} & [V/G] \end{array}, \qquad \begin{aligned} R_U &= \{(x, y, g) \in U \times U \times G : f(y) = g \cdot f(x)\} \\ &= \bigsqcup_{g \in G} f^{-1}(V^g) \end{aligned}.$$

The component $f^{-1}(V^{g_0}) \subset R_U$ is the desired neighborhood W consisting of points (x, y, g_0). □

2.3. **The difference morphism.** The discussions so far were purely formal about the finite flat presentation $p : A \to B$. Let us now use the fact that A is an abelian variety.

Definition 2.13. Write $A_\cdot := (A, 0)$. The *difference* morphism $\delta : R \to A_\cdot$ is the composition

$$\delta : R \xrightarrow{\text{forget}} A \times A \xrightarrow{-} A_\cdot, \qquad (x, y, g) \mapsto y - x.$$

It is neither finite nor flat. We denote the restriction of the morphism $- : A \times A \to A_\cdot$ to $\bar{R}$ by

$$\bar{\delta} : \bar{R} \to A_\cdot.$$

Remark 2.14. The variety A does not come equipped with a preferred choice of zero, so it has no canonical addition operation (i.e., A is an abelian torsor rather than an abelian variety). The notation $A_\cdot = (A, 0)$ emphasizes that A is equipped with the choice of a canonical zero, so that we can consider its addition morphism $+ : A_\cdot \times A_\cdot \to A_\cdot$. Notice that the subtraction and translation morphisms

$$- : A \times A \to A_\cdot, \qquad + : A_\cdot \times A \to A$$

are always well-defined even without fixing a zero of A.

This somewhat pedantic and annoying notation will be justified later. It will be important for us to distinguish the role of the targets of $s, t : R \to A$ and $\delta : R \to A_\cdot$. Such notation $A_\cdot$ will visually tell us the space we are working on. The discussion will be quite confusing without it.

Lemma 2.15. $I_A = \delta^{-1}(0)$.

Proof. Recall the description $I_A = \{(x, x, g) \in R\}$ in R. □

Proposition 2.16. *There exists a commutative diagram*[6]

$$\begin{array}{ccc} R \times_A R & \xrightarrow{c} & R \\ \downarrow{\scriptstyle \delta \times \delta} & & \downarrow{\scriptstyle \delta} \\ A_\cdot \times A_\cdot & \xrightarrow{+} & A_\cdot \end{array}.$$

In other words, $\delta \circ c = \delta \circ p_1 + \delta \circ p_2$.

Proof. This follows from the direct computation

$$\begin{aligned} (\delta \circ c)\Big((x, y, g), (y, z, h)\Big) = \delta(x, z, h \circ g) &= z - x \\ &= (y - x) + (z - y) = \delta(x, y, g) + \delta(y, z, h). \end{aligned}$$ □

[5] The paper used the notation ψ (with analytic neighborhoods) in AI::Lemma 11.5.

[6] This is tacitly mentioned in p.94 of the paper.

3. Fano property of the coarse moduli space

In this section, we study the difference map $\delta : R \to A.$ under the condition in Theorem 1.2(i). Recall from Proposition 2.10 that if B is not a scheme, then there exists an inertial component $R_i \subset R$. The main result of this section is a stronger version of this.

Theorem 3.1. *Let $p : A \to B$ be as usual and assume Theorem 1.2(i). If B is not a scheme, then there exists an inertial component $R_i \subset R$ with $\delta(R_i) = A.$.*

Given an inertial component R_i, OpenAI's idea is to find a new inertial component R_k with $\delta(R_i) \subsetneq \delta(R_k)$ and repeat this until $\delta(R_i) = A.$. The new inertial component R_k will be a groupoid-composition of R_i with an auxiliary irreducible component R_j. The auxiliary component R_j will be found nearby the identity component R_0 using local geometry.

3.1. **Image of the difference map.** Theorem 1.2(i) assumes B^{sch} is a Fano variety, and in particular projective. In this subsection, we will see the projectiveness of B^{sch} already deduces an interesting property of $\delta : R \to A.$.

Fix any ample line bundle on B^{sch} and consider its pullback to A via the finite morphism $A \to B \to B^{sch}$. This yields an ample line bundle P on A with the condition

$$s^*P = t^*P \quad \text{as line bundles on } R. \tag{3.2}$$

Recall that the *polarization* (with respect to P) is a homomorphism of abelian varieties

$$\lambda : A. \to \check{A}., \qquad a \mapsto [t_a^*P \otimes P^{-1}],$$

where $t_a : A \to A$ is a translation automorphism by $a \in A.$, and $\check{A}. = \mathrm{Pic}^\circ_A$ is the neutral Picard scheme of A. Given an abelian subvariety $D \subset A.$, its *complementary abelian subvariety* $D^\perp \subset A.$ is

$$D^\perp := \ker\left(A. \xrightarrow{\lambda} \check{A}. \twoheadrightarrow \check{D}\right)^\circ,$$

where the superscript $-^\circ$ means taking the identity component of an algebraic group. Notice the equality $(D^\perp)^\perp = D$.

The following is the main result of this section.

Proposition 3.3 (AI::Lemma 11.6)**.** *Let $p : A \to B$ be as usual and assume B^{sch} is projective. Then every irreducible component $R_i \subset R$ satisfies the following.*

1. *The closed subvariety $\delta(R_i) \subset A.$ is of the form*
$$\delta(R_i) = D_i + \tau_i,$$
where $D_i \subset A.$ is an abelian subvariety and $\tau_i \in A.$ is a torsion point.
2. *Set $K_i = D_i^\perp$ and define a free K_i-action on $A \times A$ by*
$$k \cdot (x, y) = (k + x, k + y).$$
Then $\bar{R}_i = \mathrm{forget}(R_i) \subset A \times A$ is closed under the K_i-action.

Corollary 3.4. *In Proposition 3.3, assume further either of the following:*

(i) *R_i is an inertial component.*
(ii) *$R_i \cap R_0 \neq \varnothing$.*

Then $\delta(R_i) \subset A.$ is an abelian subvariety.

Proof. The condition (i) (resp. (ii)) implies $R_i \cap I_A \neq \varnothing$, or equivalently $0 \in \delta(R_i)$. Hence the claim follows from Proposition 3.3(1). □

The proof of the proposition depends on the following key lemma.[7]

Lemma 3.5. *Let $s : F \to A$ be a morphism from a proper connected variety F to an abelian variety A. Then*

(1) *The pullback homomorphism $s^* : \check{A}. \to \mathrm{Pic}_F$ satisfies*

$$\dim \ker s^* \le \dim A - \dim s(F).$$

Equality holds iff $s(F) \subset A$ is an abelian subtorsor (translation of an abelian subvariety).

(2) *The same inequality holds for $\ker\left(s^* \circ \lambda : A. \to \mathrm{Pic}_F\right)$, and the equality holds iff*

$$s(F) = K + \sigma \qquad \textit{for} \quad K = \Big(\ker(s^* \circ \lambda)^\circ\Big)^\perp \subset A., \quad \sigma \in A.$$

Proof. (1) A proper connected variety over an algebraically closed field of characteristic 0 admits the notion of the Albanese variety [Con17; LS24]. This factorizes s into

$$\begin{array}{ccccc} F & & \xrightarrow{\quad s \quad} & & A \\ & \searrow^{\mathrm{alb}} & & \nearrow_{i} & \\ & & \mathrm{Alb}_F \xrightarrow{s'} H & & \end{array},$$

where $H \subset A$ is the image of $\mathrm{Alb}_F \to A$, an abelian subtorsor. Since alb^* is injective on neutral Picard schemes [Con17; LS24], we have

$$(\ker s^*)^\circ = \ker(i^* : \check{A}. \twoheadrightarrow \check{H}.). \tag{3.6}$$

This shows the dimension estimate $\dim \ker s^* = \dim A - \dim H \le \dim A - \dim s(F)$. When the equality holds, this mans $s(F) = H$ is an abelian subtorsor.

(2) The dimension inequality is immediate because $\lambda : A. \to \check{A}.$ is an isogeny. When the equality holds, we have $s(F) = H$ as above. But (3.6) shows $\ker(s^* \circ \lambda)^\circ = \ker(i^* \circ \lambda)^\circ = H.^\perp$, where the latter equality is just the definition of $H.^\perp$. This shows $H.$ is the complement of $\ker(s^* \circ \lambda)^\circ$ as claimed. □

Proof of Proposition 3.3. To ease notation, let us drop the subscript i throughout the proof. Hence R means R_i, and the diagram

$$\begin{array}{ccc} R & \xrightarrow{\text{forget}} & A \times A \\ & \searrow^{\delta} & \downarrow - \\ & & A. \end{array}$$

is the restriction of the original diagram to the irreducible component R_i.

(1) The assertion is trivial when δ is surjective, so assume $\delta(R)$ has dimension $r < n$. Over a general point $a \in \delta(R) \subset A.$, the fiber $R_a = \delta^{-1}(a)$ has pure dimension $n - r$. Take a connected component $F \subset R_a$ and consider the restricted morphisms

$$s_F : F \to A, \qquad t_F : F \to A \qquad \text{such that} \quad t_F = t_a \circ s_F.$$

Now (3.2) induces the equality in Pic_F:

$$(s_F^* \circ \lambda)(a) = s_F^*(t_a^* P \otimes P^{-1}) = t_F^* P \otimes s_F^* P^{-1} = \mathcal{O}_F.$$

[7] This is extracted from AI::Lemma 11.6.

That is, we get $a \in \ker(s_F^* \circ \lambda)$ for every connected component $F \subset R_a$. Since s_F is finite, Lemma 3.5(2) shows $\dim \ker(s_F^* \circ \lambda) \leq n-(n-r) = r$. Varying the connected component $F \subset R_a$, the general point $a \in A.$ is contained in the subgroup

$$D_a \coloneqq \bigcap_{F \subset R_a} \ker\left(s_F^* \circ \lambda\right) \qquad \text{of dimension } \leq r.$$

Therefore, there exists a Zariski open subset $U \subset \delta(R)$ that is contained in the union of proper subgroups

$$U \subset \bigcup_{a \in U} D_a, \qquad D_a \subsetneq A. \text{ is a subgroup of dimension at most } r < n.$$

But there exist only countably many proper subgroups of $A.$. This forces $U \subset D_a$ for a single $a \in U$, and therefore $\delta(R) \subset D_a$.[8] Since any subgroup of $A.$ is of the form $D \times \pi_0$ for an abelian subvariety $D \subset A.$ and finite subgroup $\pi_0 \subset A.$, we have $\delta(R) \subset D + \tau$ for some torsion point $\tau \in A.$ and an abelian variety $D.$ The sequence of dimension estimates

$$r = \dim \delta(R) \leq \dim D \leq r$$

is now all equalities. This concludes $\delta(R) = D + \tau$ and $\dim D = r$.

(2) By the equality assertion of Lemma 3.5(2), for every general point $a \in \delta(R) = D + \tau$, the fiber

$$\bar{R}_a = \bar{\delta}^{-1}(a) \cong s(R_a) \subset A$$

is a finite union of abelian subtorsors of the form $K + \sigma$ for $K = D^\perp$. This shows that $\bar{\delta}^{-1}(U)$ is closed under the given K-action on $A \times A$. Since $\bar{R}$ is irreducible, such a K-action on its Zariski open subset uniquely extends to a K-action on the whole $\bar{R}$. Note that this automatically means K acts fiberwise on $\bar{\delta} : \bar{R} \to A.$. □

3.2. **Composition of irreducible components.** Assume further that B^{sch} is a Fano variety, and let us begin the proof of Theorem 3.1 following §11.4 of the paper.

Use Proposition 2.10 to pick an inertial component $R_i \subset R$. If $\delta(R_i) = A.$ then we are done, so assume that (Corollary 3.4)

$$D = \delta(R_i) \subset A., \qquad K = D^\perp \subset A. \tag{3.7}$$

are abelian subvarieties of positive dimensions r and $n-r$. The strategy will be to find a new inertial component $R_k \subset R$ whose difference $\delta(R_k)$ is strictly larger than D. To do so, in Lemma 3.9, we will first find a suitable non-identity component $R_j \subset R$ adjacent to the identity component $(R_0 \cap R_j \neq \varnothing)$. In Lemma 3.11, the new inertial component R_k will be obtained as the *composition* of R_i and R_j using the groupoid structure of R.

Let us first make the following notation explicit.

Definition 3.8. Let $K \subset A.$ be an abelian subvariety. For each closed subvariety $X \subset A$, we define $K + X \subset A$ as the image of $K \times X \subset A. \times A$ under the translation morphism $+ : A. \times A \to A$. Similarly, for $Z \subset A \times A$ we define $K + Z \subset A \times A$ under the diagonal $A.$-action on $A \times A$.

Lemma 3.9 (AI::Lemma 11.7)**.** *There exist a non-identity component R_j together with a prime Weil divisor $E_j \subset R_j$ satisfying the following three conditions:*

(a) *$\delta(R_j) \subset A.$ is an abelian subvariety, and is not contained in D.*

[8] This argument is reminiscent of Kawamata's proof of his [Kaw81, Theorem 13].

(b) *R_j intersects with R_0, and E_j is an irreducible component of $R_0 \cap R_j$. In particular, $s : R \to A$ induces an isomorphism $E_j \cong s(E_j) \subset A$.*
(c) *$K + s(E_j) = A$.*

Proof. Let us first construct a divisor E_j in the identity component R_0. Let $T \subset A$ be the ramification divisor of $p^{sch} : A \to B \to B^{sch}$ arising in the Riemann–Hurwitz formula:

$$K_A (= \mathcal{O}_A) \sim_{\mathbb{Q}} (p^{sch})^* K_{B^{sch}} \otimes \mathcal{O}_A(T).$$

Then T is ample because $K_{B^{sch}}$ is anti-ample and p^{sch} is finite. This in particular implies $K+T = A$ (otherwise we have $K + T = T$, violating the ampleness of T). Take $E_j \subset T$ to be any prime component such that (c) $K + E_j = A$. We may consider E_j as a divisor in R_0 via an isomorphism $s : R_0 \to A$.

Let us now find the component R_j. By Lemma 2.11, the morphism $s : R \to A$ is ramified along $E_j \subset R_0 \subset R$ with locally $e > 1$ number of branches. Take any non-identity component R_j containing some of these branches. Then (b) $E_j \subset R_j \cap R_0$ is satisfied.

It only remains to show (a) $\delta(R_j) \not\subset D$. Corollary 3.4 shows $\delta(R_j) = D_j \subset A$. is an abelian subvariety. Assume on the contrary that we have $D_j \subset D$. Then this implies the opposite inclusion $K_j = D_j^{\perp} \supset K$ and hence shows that $\bar{R}_j$ is closed under the K-action by Proposition 3.3(2). But $\bar{R}_0 = \Delta_A \subset A \times A$ is closed under the K-action as well, so their intersection $\bar{R}_j \cap \bar{R}_0$ is closed under the K-action. This means our irreducible component $E_j \subset R_0 = \bar{R}_0$ is also closed under the K-action, contradicting the condition (c) $K + E_j = A$. □

Lemma 3.10 (AI::Lemma 11.8)**.** *Given $E_j \subset R_j$ in Lemma 3.9, there exists a prime Weil divisor $E_i \subset R_i$ satisfying the following three conditions:*

(a) *$\delta(E_i) = D$.*
(b) *$t(E_i) = s(E_j)$ for $t : R_i \to A$ and $s : R_j \to A$.*
(c) *E_i contains a point of the form (x, x, g) with $g \neq \mathrm{id}$.*

Proof. Since R_i is inertial, we can take any point $e_0 = (x_0, x_0, g_0) \in R_i$ with $g_0 \neq \mathrm{id}$. Our first goal is to find another point

$$e_1 = (x_1, x_1, g_1) \in R_i \qquad \text{with} \quad x_1 \in s(E_j) \ \text{ and } \ g_1 \neq \mathrm{id}\,.$$

By the condition $K + s(E_j) = A$ in Lemma 3.9(c), there exists $k \in K$ such that $x_1 := k + x_0 \in s(E_j)$. Proposition 3.3(2) constructed a K-action on $\bar{R}_i$, and this action lifts to a K-action on its normalization Z_i by Lemma 2.2. Take any lift $\tilde{e}_0 \in Z_i$ and its K-orbit $K \cdot \tilde{e}_0 \subset Z_i$. Its image in R_i passes through both $e_0 = (x_0, x_0, g_0)$ and another point of the form $e_1 = (x_1, x_1, g_1)$. Now $g_0 \neq \mathrm{id}$ implies $g_1 \neq \mathrm{id}$ by Lemma 2.8.

Since $t : R_i \to A$ is finite surjective (Lemma 2.1), we may take any irreducible component $E_i \subset t^{-1}(s(E_j))$ passing through this point $e_1 = (x_1, x_1, g_1)$. It is a prime divisor in R_i with $t(E_i) = s(E_j)$. This achieves both (b) and (c).

It remains to show (a) $\delta(E_i) = D$. Recall the diagram

$$\begin{array}{ccc} R_i & \xrightarrow{\text{forget}} & \bar{R}_i \\ & \searrow^{\delta} & \downarrow{\scriptstyle \bar{\delta}} \\ & & A. \end{array}$$

and write $\bar{E}_i = \mathrm{forget}(E_i) \subset \bar{R}_i$. Let us first show $K + \bar{E}_i = \bar{R}_i$ under the K-action on $\bar{R}_i$ in Proposition 3.3(2). Taking projection of its left hand side is

$$p_2(K + \bar{E}_i) = K + t(E_i) = K + s(E_j) = A \qquad \text{by (b) and Lemma 3.9(c).}$$

Hence $K + \bar{E}_i \subset \bar{R}_i$ dominates A via the finite surjective morphism $p_2 : \bar{R}_i \to A$. But $\bar{R}_i$ is irreducible, so this is enough to show $K + \bar{E}_i = \bar{R}_i$. This claim implies the sequence of equalities

$$\delta(E_i) = \bar{\delta}(\bar{E}_i) = \bar{\delta}(K + \bar{E}_i) = \bar{\delta}(\bar{R}_i) = \delta(R_i) = D.$$

Here the second equality followed because $\bar{\delta}$ is invariant under the K-action on $\bar{R}_i$. □

Lemma 3.11. *Under Lemma 3.9–3.10, there exists a new inertial component $R_k \subset R$ satisfying $\delta(R_k) \supsetneq D$.*

Proof. Using a morphism $t : E_i \to s(E_j)$ in Lemma 3.10(b) and an isomorphism $s : E_j \to s(E_j)$ in Lemma 3.9(b), we can construct the graph morphism $\iota : E_i \hookrightarrow R_{ij}$ by $\iota(e) = (e, s^{-1}(t(e)))$ as in the following diagram:

$$\begin{array}{ccc} & & E_j \\ & & s\downarrow\cong \\ E_i & \xrightarrow{t} & s(E_j) \end{array} \quad , \qquad \begin{array}{ccccc} & & \overset{s^{-1}\circ t}{\frown} & & \\ E_i & \xrightarrow{\iota} & R_{ij} & \xrightarrow{p_2} & R_j \\ & \searrow^{\mathrm{id}} & \downarrow p_1 & & \downarrow s \\ & & R_i & \xrightarrow{t} & A \end{array} .$$

Take any irreducible component $T \subset R_{ij}$ that contains $\iota(E_i)$. Then the image

$$R_k = c(T) \subset R$$

under the composition map $c : R \times_A R \to R$ is an irreducible component by Lemma 2.4.

Let us first show that R_k is inertial. Recalling the point $e_0 = (x_0, x_0, g_0) \in E_i$ with $g_0 \neq \mathrm{id}$ in Lemma 3.10(c), we have

$$\iota(e_0) = (e_0, s^{-1}(t(e_0))) = ((x_0, x_0, g_0), (x_0, x_0, \mathrm{id})) \in T.$$

Their composition is $c((x_0, x_0, g_0), (x_0, x_0, \mathrm{id})) = (x_0, x_0, g_0) = e_0 \in R_k$, so R_k an inertial component (and shows $R_i \cap R_k \neq \varnothing$).

It remains to show $\delta(R_k) \supsetneq D$. Notice first $\delta(R_k) = (\delta \circ c)(T)$. Applying the relation $\delta \circ c = \delta \circ p_1 + \delta \circ p_2$ in Proposition 2.16 to $\iota(e) \in T$ for $e \in E_i$ yields

$$(\delta \circ c)(\iota(e)) = (\delta \circ p_1)(\iota(e)) + (\delta \circ p_2)(\iota(e)) = \delta(e) + \delta(s^{-1}(t(e))) = \delta(e).$$

Here the last equality follows from $\delta \equiv 0$ on $E_j \subset R_j$ by Lemma 3.9(b). Since $\delta(E_i) = D$ by Lemma 3.10(a), we get an inclusion $(\delta \circ c)(T) \supset D$.

On the other hand, pick a point $t \in T$ with $(\delta \circ p_2)(t) \notin D$ using the surjectivity of $p_2 : T \to R_j$ by Lemma 2.4 and $\delta(R_j) \not\subset D$ by Lemma 3.9(a). Since $(\delta \circ p_1)(t) \in \delta(R_i) = D$, this shows

$$(\delta \circ c)(t) = (\delta \circ p_1)(t) + (\delta \circ p_2)(t) \quad \notin \quad D,$$

so the inclusion is strict as desired. □

Proof of Theorem 3.1. If B is not a scheme, then we can find an inertial component $R_i \subset R$. If $\delta(R_i) = D$ is the entire A., then we are done. If not, then use Lemma 3.9–3.11 to find another inertial component $R_k \subset R$ with $\delta(R_k) \supsetneq D$. Induction completes the proof. □

4. Derivative of the presentation

This section summarizes AI::Lemma 11.4–11.5.

Theorem 4.1 (AI::Lemma 11.5)**.** *Let $p : A \to B$ be as usual and assume Theorem 1.2(ii). Then every inertial component $R_i \subset R$ satisfies*

$$\delta(R_i) \subsetneq A.$$

The theorem does not assume B^{sch} is projective; we do not need Proposition 3.3. In fact, the proof of this theorem is almost purely local on A. Besides developing a general theory, the two main ingredients are (1) interpretation of the top Chern class of a vector bundle as the vanishing locus of its regular section, and (2) translating the information on the difference morphism δ to a condition on the derivative $dp : T_A \to p^*T_B$ of the morphism p.

4.1. **Incidence correspondence and the Euler characteristic assumption.** At each point $x \in A$, the morphism $p : A \to B$ induces a derivative homomorphism

$$dp_x : T_xA \to T_{p(x)}B.$$

Again for each automorphism g of $p(x)$, the tangent space $T_{p(x)}B$ admits the g-eigenspace decomposition $T^g_{p(x)}B \oplus$ (complement). Composing dp_x with the projection $T_{p(x)}B \to T^g_{p(x)}B$ yields a linear map

$$\alpha_{(x,g)} : T_xA \xrightarrow{dp_x} T_{p(x)}B \twoheadrightarrow T^g_{p(x)}B \ . \tag{4.2}$$

In this section, we prove the following result about this map (4.2).

Proposition 4.3 (AI::Lemma 11.4)**.** *Assume Theorem 1.2(ii). Then a general vector field $v \in H^0(A, T_A)$ satisfies $\alpha_{(x,g)}(v) \neq 0$ for every $x \in A$ and every automorphism g of $p(x) \in B$.*

To understand this statement more systematically, recall the cartesian diagram between the inertia stack and stabilizer scheme

$$\begin{array}{ccc} I_A & \xrightarrow{p} & I_B \\ \downarrow j & & \downarrow j \\ A & \xrightarrow{p} & B \end{array} .$$

The pullback $p^*\Omega_B \to \Omega_A$ models the dual of the derivative dp_x above. Its restriction to the splitting injection $\Omega_{I_B} \hookrightarrow j^*\Omega_B$ is a homomorphism of vector bundles on I_A modeling the dual of (4.2)[9]

$$\check{\alpha} : p^*\Omega_{I_B} \hookrightarrow p^*j^*\Omega_B = j^*p^*\Omega_B \twoheadrightarrow j^*\Omega_A \ (\cong \mathcal{O}_{I_A}^{\oplus n}) \ .$$

Proposition 4.3 studies the property of the set

$$L = \{(x, g, v) \in I_A \times \mathbb{A}^n : \alpha_{(x,g)}(v) = 0\}$$

parametrizing the *fiberwise* kernel of α. The set admits a scheme structure as follows.

Lemma 4.4. *L is a scheme with an incidence correspondence*

$$\begin{array}{ccc} L & \xrightarrow{q} & \mathbb{A}^n \\ \downarrow & & \\ I_A & & \end{array} .$$

[9]This is the dual of the homomorphism $\alpha : j^*T_A \to p^*T_{I_B}$ defined after AI::(11.6) in p.89.

Proof. Take the cokernel of $\check{\alpha} : p^*\Omega_{I_B} \to \mathcal{O}_{I_A}^{\oplus n}$ above. It admits a surjection $\mathcal{O}_{I_A}^{\oplus n} \twoheadrightarrow \operatorname{coker}\check{\alpha}$. We can define L as the linear scheme associated to $\operatorname{coker}\check{\alpha}$:

$$L \coloneqq \operatorname{Spec}_{I_A}\Big(\operatorname{Sym}^*(\operatorname{coker}\check{\alpha})\Big) \quad \subset \quad I_A \times \mathbb{A}^n = \operatorname{Spec}_{I_A}\Big(\operatorname{Sym}^*(\mathcal{O}_{I_A}^{\oplus n})\Big).$$

This L is the *fppf dual* of $\operatorname{coker}\check{\alpha}$, so it parametrizes the fiberwise kernel $\ker\alpha_{(x,g)}$ for every point $(x,g) \in I_A$. □

Lemma 4.5. *The degeneracy locus*

$$I_{A,\leq r} \coloneqq \{(x,g) \in I_A : \operatorname{rk}\alpha_{(x,g)} \leq r\}$$

is a closed subscheme of I_A of dimension $\leq r$.

Proof. Note that $\alpha_{(x,g)}$ and $\check{\alpha}_{(x,g)}$ have the same rank, so this set is the degeneracy locus of a vector bundle homomorphism $\check{\alpha}$, see [Ful98, §14]. Hence it is a closed subscheme of I_A.

To show the dimension claim, we may show instead: $\check{\alpha}$ has a generic rank $\geq s$ along every irreducible subvariety $i : Y \hookrightarrow I_A$ of dimension s.

$$\begin{array}{ccccc} Y & \xhookrightarrow{i} & I_A & \xrightarrow{p} & I_B \\ & \searrow & \downarrow j & & \\ & & A & & \end{array}$$

Notice that $j \circ i$ is unramified because j is unramified (Lemma 2.8). Hence we have a surjective homomorphism $(j\circ i)^*\Omega_A \twoheadrightarrow \Omega_Y$. The composition

$$(p\circ i)^*\Omega_{I_B} \xrightarrow{\check{\alpha}_{|Y}} (j\circ i)^*\Omega_A \twoheadrightarrow \Omega_Y$$

is nothing but the cotangent sequence of the morphism $p \circ i$. This homomorphism is generically surjective because the finite morphism $p \circ i$ is generically unramified. Therefore, it has a generic rank s. Hence $\check{\alpha}_{|Y}$ has a generic rank $\geq s$ by the surjectivity of the second homomorphism. □

Corollary 4.6. (1) *The set $I_{A,0} = \{(x,g) \in I_A : \alpha_{(x,g)} \equiv 0\}$ is finite.*

(2) *Every general $v \in H^0(A, T_A)$ satisfies $\alpha_{(x,g)}(v) \neq 0$ for every $(x,g) \in I_{A,\geq 1}$.*

Proof. (1) Immediate from Lemma 4.5. (2) The morphism $L \to I_A$ is an $\mathbb{A}^{n-r}$-bundle over $I_{A,r}$, which has dimension $\leq r$ by Lemma 4.5. Hence every irreducible component of L has dimension $\leq n$.

Let $L_i \subset L$ be an irreducible component, and consider the projection morphism in Lemma 4.4:

$$q : L \to \mathbb{A}^n, \qquad (x,g,v) \mapsto v.$$

If $q(L_i) = \mathbb{A}^n$, then this forces L_i to be of the form

$$L_i = \{(x,g)\} \times \mathbb{A}^n \qquad \text{for} \quad (x,g) \in I_{A,0}.$$

If $q(L_i) \subsetneq \mathbb{A}^n$, then $q(L_i)$ a proper constructible subset of $\mathbb{A}^n$ by Chevalley's theorem. The union of all such proper subsets $q(L_i)$ is still proper, so its complement contains a Zariski open subset $U \subset \mathbb{A}^n$. Combining these two, we obtain the description

$$q^{-1}(v) = I_{A,0} \times \{v\} \qquad \text{for every } v \in U \subset \mathbb{A}^n,$$

which is the desired claim. □

Proof of Proposition 4.3. All the above lemmas and corollary were completely formal and the assumption in Theorem 1.2(ii) was never used. Now we prove that the given assumption implies $I_{A,0} = \varnothing$ and completes the proof of the proposition.

Take the Zariski open subset $U \subset \mathbb{A}^n$ in Corollary 4.6 and let $v \in U$ be any generic vector. We have shown $q^{-1}(v) = I_{A,0} \times \{v\}$, which means that the global section $\alpha(v)$ of $p^*T_{I_B}$ has a finite vanishing set $I_{A,0}$. Therefore, according to the geometric interpretation of the top Chern class of vector bundles [Ful98, Proposition 14.1] [Stacks, Tag 0FA9], we can show $I_{A,0} = \varnothing$ once the top Chern class of $p^*T_{I_B}$ vanishes. The assumption on the Euler characteristic of each connected component means the top Chern class of T_{I_B} vanishes already on I_B by the Gauss–Bonnet theorem in [Beh09, Proposition 1.6]. This completes the proof. □

4.2. **Local structure of the equivalence relation.** Theorem 4.1 is essentially a *local* study of the morphism $p : A \to B$. The idea is to translate the surjectivity of the difference map δ into a local property of p. This is the content of the following observation, which is elementary but crucial.

Lemma 4.7. *Let Δ be a smooth affine curve with a reference point 0, and $c : \Delta \to A \times A$ a nonconstant morphism with $c(0) = (x_0, y_0)$. Consider three compositions*

$$c_1 \coloneqq p_1 \circ c, \quad c_- \coloneqq - \circ c, \quad c_2 \coloneqq p_2 \circ c \qquad \textit{for} \qquad \begin{array}{ccccc} \Delta & \xrightarrow{c} & A \times A & \xrightarrow{p_2} & A \\ & & {\scriptstyle p_1}\downarrow & \searrow^{-} & \\ & & A & & A. \end{array}$$

Then we have an equality

$$dc_-(\partial_t) = dc_2(\partial_t) - dc_1(\partial_t) \qquad \textit{for} \quad \partial_t = \tfrac{\partial}{\partial t} \in T_0\Delta$$

under the identifications $T_{x_0}A = T_{y_0 - x_0}A_\cdot = T_{y_0}A = H^0(A, T_A)$.

Proof. The easiest way to see this over $\mathbb{C}$ is by setting $A = \mathbb{C}^n/\Gamma$. The statement is analytically local, so we can pretend Δ is a unit disc, $A = T_0A = \mathbb{C}^n$, and $x_0 = y_0 = 0$. Then the subtraction map on A coincides with that on $\mathbb{C}^n$. Now the lemma is saying nothing but $(f-g)'(0) = f'(0) - g'(0)$ for two holomorphic functions $f, g : \mathbb{C} \to \mathbb{C}^n$. □

To prove Theorem 4.1, assume on the contrary that $R_i \subset R$ is an inertial component with surjective $\delta : R_i \to A_\cdot$. Then Lemma 4.7 has the following consequence.

Proposition 4.8 (AI::Lemma 11.5)**.** *Let R_i be an inertial component with surjective $\delta : R_i \to A_\cdot$. Then for every general $v \in H^0(A, T_A)$, there exist a point $(x, x, g) \in R_i$ with $g \neq \mathrm{id}$ and a local presentation $f : (U, x) \to (V, 0)$ of p such that*

$$df_x(v) \in \mathrm{im}(\mathrm{id} - g) \quad (\subset T_0V).$$

Proof. Every tangent vector $v \in H^0(A, T_A) = T_0A_\cdot$ is realized as the direction vector $dc_-(\partial_t)$ of a morphism $c_- : \Delta \to A_\cdot$ from a smooth affine curve Δ with $c_-(0) = 0$. The surjectivity assumption on $\delta : R_i \to A_\cdot$ means that δ is generically étale. Thus, a general choice of $v \in T_0A_\cdot$ makes δ generically étale over the curve Δ. We will shrink Δ around 0 whenever needed.

Pick a point $(x_0, x_0, g_0) \in R_i$ with $g_0 \neq \mathrm{id}$, and take a connected component $F \subset R_i \cap I_A$ containing this point. Since $\delta : R_i \to A_\cdot$ is generically étale over Δ, we can lift this curve to a

rational section $c : \Delta \dashrightarrow R_i$ through F. The valuative criterion (δ is proper) fills in the missing value $c(0)$ and constructs a lifting

$$\begin{array}{ccc} & \overset{c}{\nearrow} & R_i \\ & & \downarrow\delta \\ \Delta & \xrightarrow{c_-} & A. \end{array}$$

By construction, $c(0)$ passes through the connected component F of $R_i \cap I_A$, which contains the given point (x_0, x_0, g_0) with $g_0 \neq \mathrm{id}$. Recalling Lemma 2.8, the point $c(0)$ must be of the form (x_1, x_1, g_1) with $g_1 \neq \mathrm{id}$.

Apply Lemma 2.12 to the point (x_1, x_1, g_1) to take a local presentation $f : U \to V$ of $p : A \to B$ and a local neighborhood W of R_i. Taking its image $\bar{W} = \mathrm{forget}(W) \subset U \times U$, we have the equality

$$f(y) = g_1 \cdot f(x) \qquad \text{for} \quad (x, y) \in \bar{W} \tag{4.9}$$

Denote the curve on $\bar{W}$ by $c : \Delta \to \bar{W}$ for simplicity, and write $c = (c_1, c_2)$ for $c_1, c_2 : \Delta \to U$.

$$\begin{array}{ccccc} & & \overset{c_2}{\frown} & & \\ \Delta & \xrightarrow{c} & \bar{W} & \xrightarrow{p_2} & U \\ & \underset{c_1}{\searrow} & \downarrow p_1 & & \\ & & U & & \end{array}$$

Lemma 4.7 claims $v = dc_2(\partial_t) - dc_1(\partial_t)$, so applying df and using (4.9) shows the desired claim

$$\begin{aligned} df(v) &= d(f \circ c_2)(\partial_t) - d(f \circ c_1)(\partial_t) \\ &= g \cdot d(f \circ c_1)(\partial_t) - d(f \circ c_1)(\partial_t) = (g - \mathrm{id}) \cdot d(f \circ c_1)(\partial_t). \end{aligned}$$

□

Proof of Theorem 4.1. The two conclusions Proposition 4.3 and Proposition 4.8 are incompatible. To see this, recall that $\alpha_{(x,g)}(v)$ was defined to be $dp(v) = df_x(v)$ projected down to the g-invariant subspace of $T_{p(x)}V = T_0V$. The projection homomorphism is explicitly described as the averaging map

$$T_0V \twoheadrightarrow T_0^g V, \qquad v \mapsto \tfrac{1}{d}(\mathrm{id} + g + \cdots + g^{d-1})(v),$$

where d is the order of the linear map g on T_0V. Hence Proposition 4.3 says that a general $v \in H^0(A, T_A)$ satisfies $(\mathrm{id} + g + \cdots + g^{d-1})(df_x(v)) \neq 0$ for every $(x, g) \in I_A$.

However, Proposition 4.8 constructed $(x_0, g_0) \in I_A$ such that $df_{x_0}(v) = (\mathrm{id} - g_0)(\xi)$ for some $\xi \in T_0V$. This computes

$$(\mathrm{id} + g_0 + \cdots + g_0^{d-1})(df_{x_0}(v)) = (\mathrm{id} + g_0 + \cdots + g_0^{d-1})(\mathrm{id} - g_0)(\xi) = (\mathrm{id} - g_0^d)(\xi) = 0,$$

which is the desired contradiction. □

5. Proof of main theorems

The proof of Theorem 1.2 is now immediate from the two main results of the previous sections.

Proof of Theorem 1.2. If B is not a scheme, then (i) implies Theorem 3.1 and (ii) implies Theorem 4.1. These are incompatible. □

The following proposition reduces Theorem 1.1 to Theorem 1.2.

Proposition 5.1. *Let X be a projective hyper-Kähler manifold and $f : X \to B$ a Lagrangian fibration to a smooth proper DM stack B with trivial stabilizers in codimension 1. Then*

(1) *B^{sch} is a Fano variety.*

(2) *Every (non-identity) connected component of I_B has Euler characteristic* 0.

Again, the *Euler characteristic* of a DM stack here means its stacky-Euler characteristic, which is a rational number in general as in [Beh09, §1.3]. The first item is a well-known result of Matsushita. The second item is new; OpenAI's idea was to transform another well-known theorem of Matsushita into a K-theoretic form. The rest of this note proves Proposition 5.1 and concludes.

5.1. **Euler characteristic of the inertia stack.** Let us start with some formal discussions on vector bundles on smooth, separated, and finite type DM stacks S. Given a vector bundle F of rank r on S, we define[10]

$$\wedge^*(F) = \sum_{i=0}^{r} (-1)^i \cdot \left[\bigwedge^i F\right] \qquad \text{in} \quad K^0(S).$$

Similarly, one can define $\wedge^*(\alpha)$ for any class $\alpha \in K^0(S)$.

Lemma 5.2. *Let S be a smooth separated DM stack and* ch $: K^0(S)_{\bar{\mathbb{Q}}} \to \mathrm{CH}^*(S)_{\bar{\mathbb{Q}}}$ *the Chern character homomorphism. Then*

(1) $\wedge^*(\alpha+\beta) = \wedge^*(\alpha)\cdot\wedge^*(\beta)$ *for* $\alpha, \beta \in K^0(S)$.
(2) *For each rank r vector bundle F on S, we have*

$$\mathrm{ch}\left(\wedge^*(F)\right) = (-1)^r \cdot c_r(F) \ + (\textit{terms of degree} > r),$$
$$\mathrm{ch}_0\left(\wedge^*(c\cdot[F])\right) = (1-c)^r \qquad \textit{for} \quad c \in \bar{\mathbb{Q}}.$$

Proof. (1) This is standard. (2) When F is a line bundle, both are easily deduced. When F is a vector bundle of rank $r > 1$, a standard method of Grothendieck (passing to the flag variety-bundle over S) reduces to the case when F is a direct sum of line bundles. Now use (1) and the fact that ch is a ring homomorphism. □

Let us return back to Proposition 5.1 and assume B is the smooth DM stack base of a Lagrangian fibration. Our references for some facts about inertia stacks are [Toe99, Theorem 3.15] and [Tse10, §§2.1–2.2].

Lemma 5.3 (AI::Lemma 11.3)**.** *Every rank r vector bundle E on B satisfies an equality*

$$([E]-r)^m \cdot \wedge^*(\Omega_B) = 0 \qquad \textit{for} \ \ m \gg 0 \ \ \textit{in} \ K^0(B).$$

Proof. Every proper flat Lagrangian fibration $f : X \to B$ from a quasi-projective symplectic variety X satisfies the equality $R^i f_* \mathcal{O}_X = \Omega_B^i$ by [Mat05; Sch23]. This means $[Rf_*\mathcal{O}_X] = \wedge^*(\Omega_B)$, or similarly using the projection formula

$$Rf_*([f^*E]^i) = [E]^i \cdot \wedge^*(\Omega_B) \qquad \text{in} \quad K^0(B).$$

Now the rank r vector bundle f^*E on a smooth variety X satisfies the identity (because in this case, the Chern character homomorphism is an isomorphism)

$$([f^*E]-r)^m = 0 \qquad \text{for} \ \ m > 2n \quad \text{in } K^0(X).$$

Applying Rf_* to this identity gives the desired claim. □

[10] Our notation is not standard. OpenAI followed the standard notation $\lambda_{-1}(F)$ in the context of λ-rings.

Let $S \subset I_B$ be a non-identity connected component of the inertia stack, and write $j : S \subset I_B \to B$ for the composition. Let E be a vector bundle on B and j^*E its pullback to S. Locally around a point $(z, g) \in S$, the vector bundle j^*E is g-equivariant and hence admits the eigenbundle decomposition $j^*E = \bigoplus_{i=0}^{k-1} E^{(i)}$ where k is the order of g, $E^{(i)}$ is the subbundle with eigenvalue ζ^i, and ζ is a primitive k-th root of unity. Such a decomposition patches together into

$$j^*E = \bigoplus_{i=0}^{k-1} E^{(i)} \qquad \text{on} \quad S.$$

Following [Toe99, Theorem 3.15], we define a ring homomorphism ϕ by

$$\phi : K^0(B) \to K^0(S)_{\bar{\mathbb{Q}}}, \qquad [E] \mapsto \sum_{i=0}^{k-1} \zeta^i \cdot [E^{(i)}]. \tag{5.4}$$

Composing it with the Chern character map of S is the *inertial Chern character* homomorphism

$$\mathrm{ch} \circ \phi : K^0(B) \to \mathrm{CH}^*(S)_{\bar{\mathbb{Q}}}.$$

Proof of Proposition 5.1. (1) This is a well-known result of Matsushita [Mat99, Theorem 2], which uses the global hyper-Kähler geometry of X. (2) Let S be a non-identity component of I_B, say of dimension $d < n$, with a morphism $j : S \to B$. Say the stabilizers in S have order k and ζ is a primitive k-th root of unity as above.

Start with the equality in Lemma 5.3 (for an arbitrary vector bundle E on B). Applying the homomorphism ϕ (5.4) to this equality and using Lemma 5.2(1), we obtain

$$\Big(\phi(E) - r\Big)^m \cdot \prod_{i=0}^{k-1} \Big(\wedge^* (\zeta^i \cdot [\Omega_B^{(i)}])\Big) = 0 \qquad \text{in} \quad K^0(S)_{\bar{\mathbb{Q}}}.$$

Now subsequently apply the Chern character to it. Lemma 5.2(2) computes

$$\begin{aligned} \mathrm{ch}\Big(\wedge^* (\Omega_B^{(0)})\Big) = \mathrm{ch}\Big(\wedge^* (\Omega_S)\Big) &= (-1)^d \cdot c_d(\Omega_S) = c_d(T_S), \\ \mathrm{ch}_0\Big(\wedge^* (\zeta^i \cdot [\Omega_B^{(i)}])\Big) &\neq 0 \qquad \text{for} \quad 0 < i < k. \end{aligned}$$

Furthermore, we have a direct computation

$$\mathrm{ch}_0\Big(\phi(E) - r\Big) = \sum_{i=0}^{k-1} r_i \cdot \zeta^i - r \qquad \text{for} \quad r_i = \mathrm{rk}\, E^{(i)}.$$

Because $\sum_i r_i = r$ is the rank of E, this Chern class is nonzero iff $r_0 < r$. Simply take a vector bundle E on B satisfying this property; for example, the paper used $E = T_B$ in AI::Lemma 11.3–11.4 because $T_S = T_B^{(0)}$ is a proper subbundle of T_B (same works for $E = \Omega_B$).

Combining all the above computations yields the equality AI::(11.6)

$$c_d(T_S) = 0 \qquad \text{in} \quad \mathrm{CH}^*(S)_{\bar{\mathbb{Q}}}.$$

Our argument so far never used the properness of B and holds locally on B. When B (and hence S) is proper, a prettier way of saying this is $\chi(S) = 0$. This completes the proof. □

Proposition 5.1(2) implies that *every* connected component I_B has dimension ≥ 1.[11] Consequently, B^{sch} has no isolated singularities. In fact, such a result was already proved by Müller–Xu in [MX26]. Their paper precedes OpenAI's paper and the two proofs have similarities. We invite the interested readers to check Müller–Xu's paper, which may have motivated OpenAI's approach to Proposition 5.1(2).

Theorem 5.5 ([MX26, Theorem 1.3])**.** *Let $f : X \to B$ be a proper Lagrangian fibration of a smooth quasi-projective symplectic variety X to a smooth DM stack B. Then I_B has no components with dimension 0. In particular, B^{sch} has no isolated singularities.*

Proof. The main result of their paper claims that B has no isolated singularities, but in fact the same proof concludes this slightly stronger claim. Let us quickly review their method to see this.

The statement is local on B, so we may assume $B = [V/G]$ where V is a smooth variety with a G-action fixing $0 \in V$. Taking the base change by $V \to B$ (and writing B for V for simplicity) yields a G-equivariant Lagrangian fibration $f : X \to B$ where the G-action on X is free, and that on B has a fixed point $0 \in B$. Denote by $F = X_0$ its central fiber, which admits a free G-action. Matsushita's isomorphism $R^i f_* \mathcal{O}_X = \Omega_B^i$ is interpreted as an identification $H^i(F, \mathcal{O}_F) = \wedge^i(T_0^* B)$, which is also G-equivariant in this case.

Let $g(\neq \mathrm{id}) \in G$ be any non-identity element. Then g acts on F freely. A version of the holomorphic Lefschetz fixed point formula in [BFQ79] (see also [MX26, Theorem 2.5]) says that the g-action on $H^*(F, \mathcal{O}_F) = \wedge^*(T_0^* B)$ satisfies $\det(g - \mathrm{id}) = 0$. This means that the g-action on $T_0^* B$ has a nontrivial g-eigenspace with eigenvalue 1, or equivalently the g-action on B has a fixed locus B^g of dimension ≥ 1. This means every connected component of I_B has dimension ≥ 1. □

Müller–Xu presented some examples of Lagrangian fibrations $f : X \to B$ (with non-proper B) where I_B has components of dimension ≥ 1. As shown in the proof of Proposition 5.1(2), their tangent bundles must have trivial top Chern classes.

References

[AV02] Dan Abramovich and Angelo Vistoli. "Compactifying the space of stable maps". In: *J. Amer. Math. Soc.* 15.1 (2002), pp. 27–75.

[Beh09] Kai Behrend. "Donaldson-Thomas type invariants via microlocal geometry". In: *Ann. of Math. (2)* 170.3 (2009), pp. 1307–1338.

[BFQ79] Paul Baum, William Fulton, and George Quart. "Lefschetz-Riemann-Roch for singular varieties". In: *Acta Math.* 143.3-4 (1979), pp. 193–211.

[Con17] Brian Conrad. *Albanese variety over non-perfect fields*. MathOverflow. https://mathoverflow.net/q/261032 (version: 2017-01-31). 2017.

[FMN10] Barbara Fantechi, Etienne Mann, and Fabio Nironi. "Smooth toric Deligne-Mumford stacks". In: *J. Reine Angew. Math.* 648 (2010), pp. 201–244.

[Ful98] William Fulton. *Intersection theory*. Second. Vol. 2. A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, 1998.

[Kaw81] Yujiro Kawamata. "Characterization of abelian varieties". In: *Compositio Math.* 43.2 (1981), pp. 253–276.

[KM97] Seán Keel and Shigefumi Mori. "Quotients by groupoids". In: *Ann. of Math. (2)* 145.1 (1997), pp. 193–213.

[Liu26] Zhiyu Liu. "Smoothness criterion for Fano orbifolds and bases of Lagrangian fibrations". 2026. arXiv: 2610.20496v1.

[11] See right after AI::(11.6) in p.89.

[LS24] Bruno Laurent and Stefan Schröer. "Para-abelian varieties and Albanese maps". In: *Bull. Braz. Math. Soc. (N.S.)* 55.1 (2024), Paper No. 4, 39.

[Mat05] Daisuke Matsushita. "Higher direct images of dualizing sheaves of Lagrangian fibrations". In: *Amer. J. Math.* 127.2 (2005), pp. 243–259.

[Mat99] Daisuke Matsushita. "On fibre space structures of a projective irreducible symplectic manifold". In: *Topology* 38.1 (1999), pp. 79–83.

[MX26] Niklas Müller and Zheng Xu. "Lagrangian fibrations onto varieties with isolated quotient singularities". In: *J. Reine Angew. Math.* 832 (2026), pp. 297–307.

[Ols06] Martin Olsson. "Hom-stacks and restriction of scalars". In: *Duke Math. J.* 134.1 (2006), pp. 139–164.

[Open26] OpenAI. *Projective-space bases of Lagrangian fibrations*. OpenAI Math Release preprint. Link to Github. 2026.

[Sch23] Christian Schnell. "Hodge theory and Lagrangian fibrations on holomorphic symplectic manifolds". 2023. arXiv: 2303.05364v2.

[Stacks] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu. 2024.

[Toe99] B. Toen. "Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford". In: *K-Theory* 18.1 (1999), pp. 33–76.

[Tse10] Hsian-Hua Tseng. "Orbifold quantum Riemann-Roch, Lefschetz and Serre". In: *Geom. Topol.* 14.1 (2010), pp. 1–81.

Department of Mathematics, Columbia University, New York, NY 10027, USA

HCMC, Korea Institute for Advanced Study, Seoul, South Korea
Email address: yk3029@columbia.edu